Math 3210 § 3. **First Midterm Exam** Name: Solutions
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1. *Prove that for every natural number n the quantity $7^{2n} - 1$ is divisible by 48.*

   We argue by induction.

   Base Case. When $n = 1$ the quantity is $7^{2n} - 1 = 7^2 - 1 = 48$ which is divisible by 48.

   Induction Case. We assume that for some natural number n the quantity is divisible by 48 so there is and integer k so that $7^{2n} - 1 = 48k$. To prove it for $n + 1$ we have

   $$7^{2(n+1)} - 1 = 7^2 \cdot 7^{2n} - 1 = 49 \cdot 7^{2n} - 1.$$

   Using the induction hypothesis $7^{2n} = 48k + 1$,

   $$7^{2(n+1)} - 1 = 49 \cdot \big(48k + 1\big) - 1 = 49 \cdot 48k + 48 = 48(49k + 1),$$

   which means that $7^{2(n+1)} - 1$ is divisible by 48 and the induction step is proven.

   Since both the base and induction cases hold, by induction we conclude that $7^{2n} - 1$ is divisible by 48 for all $n \in \mathbb{N}$.

2. *Recall the axioms of a field $(\mathcal{F}, +, \times)$. For any $x, y, z \in \mathcal{F}$,*

[A1.]	(Commutativity of Addition)	$x + y = y + x.$
[A2.]	(Associativity of Addition)	$x + (y + z) = (x + y) + z.$
[A3.]	(Additive Identity)	$(\exists\, 0 \in \mathcal{F})\, (\forall\, t \in \mathcal{F})\ 0 + t = t.$
[A4.]	(Additive Inverse)	$(\exists\, -x \in \mathcal{F})\ x + (-x) = 0.$
[M1.]	(Commutativity of Multiplication)	$xy = yx.$
[M2.]	(Associativity of Multiplication)	$x(yz) = (xy)z.$
[M3.]	(Multiplicative Identity)	$(\exists\, 1 \in \mathcal{F})\ 1 \neq 0$ and $(\forall\, t \in \mathcal{F})\ 1t = t.$
[M4.]	(Multiplicative Inverse)	If $x \neq 0$ then $(\exists\, x^{-1} \in \mathcal{F})\ (x^{-1})x = 1.$
[D.]	(Distributivity)	$x(y + z) = xy + xz.$

   Using only the field axioms, show that for any $a, b \in \mathcal{F}$ such that $a \neq 0$ and $b \neq 0$ we have

   $$a + b = (a^{-1} + b^{-1})(ab). \tag{1}$$

   Justify every step of your argument using just the axioms listed here.

   [Hint: the first line of your argument must not be "$a + b = (a^{-1} + b^{-1})(ab).$"]

   Starting from the right side, we deduce a sequence of equalities that are justified by the axioms and end up with the left side. Since $a \neq 0$ and $b \neq 0$ there are multiplicative inverses a^{-1} and b^{-1} in $\mathcal{F}$ by the existence of a multiplicative inverse (M4) and so the right side exists.

$(a^{-1}+b^{-1})(ab)$	Start with the right hand member.
$= (ab)(a^{-1}+b^{-1})$	Commutivity of multiplication. (M1)
$= (ab)a^{-1}+(ab)b^{-1}$	Distributivity. (D)
$= (ba)a^{-1}+(ab)b^{-1}$	Commutivity of multiplication. (M1)
$= b(aa^{-1})+a(bb^{-1})$	Associativity of multiplication. (M2)
$= b(a^{-1}a)+a(b^{-1}b)$	Commutativity of multiplication. (M1)
$= b(1)+a(1)$	Multiplicative inverse. (M4)
$= (1)b+(1)a$	Commutivity of multiplication. (M1)
$= b+a$	Multiplicative identity. (M3)
$= a+b$	Commutivity of addition. (A1)

Since all expressions are equal, we conclude that the left side of (1) equals the right.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

(a) *Let $f : A \to B$ be a function and $E, F \subset B$. If $f^{-1}(E) = f^{-1}(F)$ then $E = F$.*
FALSE. Let $f : \mathbf{R} \to \mathbf{R}$ be the constant function $f(x) = 0$. Let $E = \{0\}$ and $F = \mathbf{R}$. Then $f^{-1}(E) = \mathbf{R} = f^{-1}(F)$ but $E \neq F$.

(b) *If $f : \mathbf{R} \to \mathbf{R}$ is function of the real numbers such that $f(x) \neq f(y)$ implies $x \neq y$ for all $x, y \in \mathbf{R}$, then f is* one to one.
FALSE. Let $f : \mathbf{R} \to \mathbf{R}$ be the constant function $f(x) = 1$. Then for every $x, y \in \mathbf{R}$, $f(x) \neq f(y)$ is never true so that the implication $f(x) \neq f(y) \implies x \neq y$ is always true. However f is not one-to-one since $f(0) = f(1) = 1$. In fact, the condition is true for all functions, one to one or not. The contrapositive is: $x = y$ implies $f(x) = f(y)$ for all $x, y \in \mathbf{R}$.

(c) *There is a real number $x \in \mathbf{R}$ such that $x^3 = 2$.*
TRUE. Consider the set $E = \{x \in \mathbf{R} : x^3 < 2\}$. Since $0^3 < 2$, we have $0 \in E$ so E is nonempty. Since $x > 2$ implies $x^3 > 2^3 = 8 > 2$, then no $x > 2$ is in E. Hence E is bounded above by 2. By the Completeness of $\mathbf{R}$, there is a least upper bound of E, namely, the real number $m = \operatorname{lub} E$. m satisfies $m^3 = 2$.
This is all that has to be included in the argument for full credit. For completeness sake, here is an argument that $m^3 = 2$. Assume for contradiction that $m^3 \neq 2$.
In case $m^3 < 2$, let $\varepsilon = \min\{1, .01(2 - m^3)\} > 0$. Let $t = m + \varepsilon$. Then

$$t^3 = m^3 + 3m^2\varepsilon + 3m\varepsilon^2 + \varepsilon^3 \le m^3 + 12\varepsilon + 6\varepsilon + \varepsilon$$

since $\varepsilon \le 1$ and $m \le 2$. It follows that

$$t^3 \le m^3 + 19\varepsilon \le m^3 + .19(2 - m^3) = .81m^3 + .19 \cdot 2 < 2.$$

Thus we have reached a contradiction in this case: there is a number t larger than m which satisfies $t^3 < 2$, hence $t \in E$, but is not bounded above by m.
In case $m^3 > 2$, let $\varepsilon = \min\{1, .01(m^3 - 2)\} > 0$. Since m is the least upper bound, there is a $t \in E$ such that $t > m - \varepsilon$. For this t,

$$t^3 > (m - \varepsilon)^3 = m^3 - 3m^2\varepsilon + 3m\varepsilon^2 - \varepsilon^3 \ge m^3 - 12\varepsilon - \varepsilon$$

since $\varepsilon \le 1$ and $0 \le m \le 2$. It follows that

$$t^3 > m^3 - 13\varepsilon \ge m^3 - .13(m^3 - 2) = .87m^3 + .13 \cdot 2 > 2.$$

Thus we have reached a contradiction in this case also: the number $t \in E$ but $t^3 > 2$.

4. *Recall that the rational numbers are defined to be the set of equivalence classes* $\mathbb{Q} = S/\sim$ *where* $S = \left\{ \frac{a}{b} : a, b \in \mathbb{Z},\ b \neq 0 \right\}$ *is the set of symbols (pairs of integers) and the symbols are equivalent if they represent the same fraction* $\frac{a}{b} \sim \frac{c}{d}$ *iff* $ad = bc$. *We denote the equivalence class, the "fraction,"* $\left[\frac{a}{b}\right]$ *to distinguish it from a "symbol"* $\frac{a}{b}$ *from* S. *Addition and multiplication of rationals, for example, is defined on equivalence classes by*

$$\left[\frac{m}{n}\right] + \left[\frac{r}{t}\right] = \left[\frac{mt + nr}{nt}\right], \qquad \left[\frac{m}{n}\right] \cdot \left[\frac{r}{t}\right] = \left[\frac{mr}{nt}\right].$$

(a) *Show that the function* $f : \mathbb{Q} \to \mathbb{Q}$ *is well defined, where* $f\left(\left[\frac{p}{q}\right]\right) = \left[\frac{p^2}{p^2 + q^2}\right]$.

We have to show that if we take different representatives from the class of the argument, then the function values give the same class. In other words,
if $\frac{p}{q} \sim \frac{p'}{q'}$ then $\frac{p^2}{p^2+q^2} \sim \frac{(p')^2}{(p')^2 + (q')^2}$.
$\frac{p}{q} \sim \frac{p'}{q'}$ means that $pq' = qp'$. Now, using this equation, we find

$$\begin{aligned} p^2\big((p')^2 + (q')^2\big) &= p^2(p')^2 + p^2(q')^2 = p^2(p')^2 + (pq')^2 \\ &= p^2(p')^2 + (p'q)^2 = p^2(p')^2 + q^2(p')^2 = \big(p^2 + q^2\big)(p')^2. \end{aligned}$$

Thus $\frac{p^2}{p^2+q^2} \sim \frac{(p')^2}{(p')^2 + (q')^2}$ follows, as to be shown.

(b) *Is there a rational number* $r > 0$ *such that* $r^2 = r + 1$*? Explain why or why not.*

No. Suppose r were rational for contradiction, then as in the theorem from Taylor, we show that a rational solution must be integral. Let $r = \frac{p}{q}$ where $p, q \in \mathbb{N}$ and have no common factors. Then

$$\frac{p^2}{q^2} = \frac{p}{q} + 1$$

or

$$p^2 = pq + q^2.$$

Thus if $s > 1$ is a prime factor of q then since s divides the right side, it divides the left side p^2. As s is prime, it must divide p. This is a contradiction if $q \neq 1$ because then both p and q would have s as a common factor. If $q = 1$ then r is an integer. We show an integer cannot solve the equation. The equation becomes $f(r) = 0$ where

$$f(r) = r^2 - r - 1 = \left(r - \frac{1}{2}\right)^2 - \frac{5}{4}.$$

and we have completed the square to see that f is increasing for $x \geq \frac{1}{2}$. We have $f(1) = -1$, and $f(n) \geq f(2) = 1$ for $n \geq 2$ so there are no integral zeros. Hence there is no integral solution.

One could also apply the theorem in Taylor's text to conclude that any rational solution of $r^2 = r + 1$ is an integer and rule out integers as we have done.

5. *Let* $E \subset \mathbb{R}$ *be a set of real numbers given by* $E = \left\{ \dfrac{5n - n^2}{5 + n^2} : n \in \mathbb{N} \right\}$. *Define:* ℓ *is the greatest lower bound of* E. *Find the greatest lower bound of* E *and prove your assertion.*

ℓ is the *greatest lower bound* of E means that ℓ is a real number which (1) is a lower bound: $(\forall t \in E)\ \ell \le t$; and (2) no larger number is a lower bound: $(\forall y > \ell)(\exists t \in E)\ t < y$.

We claim $\operatorname{glb} E = -1$. To see (1), we observe that the n-th term

$$x_n = \frac{5n - n^2}{5 + n^2} = \frac{5n + 5 - 5 - n^2}{5 + n^2} = \frac{5 + 5n}{5 + n^2} - 1 \ge 0 - 1 = -1$$

for all $n \in \mathbb{N}$ since $5 + 5n \ge 0$ and $5 + n^2 > 0$. Thus $\ell = -1$ is a lower bound.

To see (2), choose any number $y > -1$. By the Archimedean Property, there is an integer $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \frac{y + 1}{10}.$$

For this n the element $x_n \in E$ satisfies

$$x_n = \frac{5 + 5n}{5 + n^2} - 1 \le \frac{5n + 5n}{n^2} - 1 = \frac{10}{n} - 1 < (y + 1) - 1 = y$$

since we have increased the numerator and decreased the denominator. Thus y is not a lower bound since there is an $x_n \in E$ such that $x_n < y$. (1) and (2) imply $\operatorname{glb} E = -1$.