

1. Let $a_n = \frac{n + (-1)^n}{n + 2}$. Define: $L = \lim_{n \rightarrow \infty} a_n$. Find L using limit laws. Prove using just your definition that $L = \lim_{n \rightarrow \infty} a_n$.

The limit $L = \lim_{n \rightarrow \infty} a_n$ is a number such that for all $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|a_n - L| < \varepsilon \quad \text{whenever } n \in \mathbf{N} \text{ satisfies } n > N.$$

Using the quotient, sum and limit rules

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \frac{n + (-1)^n}{n + 2} = \lim_{n \rightarrow \infty} \frac{1 + \frac{(-1)^n}{n}}{1 + \frac{2}{n}} = \frac{\lim_{n \rightarrow \infty} \left(1 + \frac{(-1)^n}{n}\right)}{\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)} \\ &= \frac{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{(-1)^n}{n}}{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{2}{n}} = \frac{1 + 0}{1 + 0} = 1. \end{aligned}$$

To prove $L = 1$, choose $\varepsilon > 0$. Let $N = \frac{3}{\varepsilon}$. For any $n \in \mathbf{N}$ such that $n > N$ we have

$$\begin{aligned} |a_n - L| &= \left| \frac{n + (-1)^n}{n + 2} - 1 \right| = \left| \frac{n + (-1)^n - (n + 2)}{n + 2} \right| = \frac{|(-1)^n - 2|}{n + 2} \\ &\leq \frac{|(-1)^n| + |2|}{n + 2} = \frac{3}{n + 2} < \frac{3}{N + 2} < \frac{3}{N} = \varepsilon. \end{aligned}$$

2. Let $a \in \mathbf{R}$ be a point and $f : \mathbf{R} \rightarrow \mathbf{R}$ be a continuous function such that $f(a) > 0$. Then there is an open interval $I \subset \mathbf{R}$ containing a such that $f(x) > 0$ for all $x \in I$.

Let $\varepsilon = f(a) > 0$. By continuity at a there is a $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \quad \text{whenever } x \in (a - \delta, a + \delta).$$

It follows that whenever $x \in (a - \delta, a + \delta)$,

$$f(x) = f(a) - [f(a) - f(x)] \geq f(a) - |f(a) - f(x)| > f(a) - \varepsilon = f(a) - f(a) = 0.$$

Thus f is positive in the interval $I = (a - \delta, a + \delta)$.

3. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

(a) STATEMENT: There is $z \in \mathbf{R}$ such that $f(z) = 0$, where $f(x) = x^4 - 2x^2 + 4x + 3$.

TRUE: $f(x)$ is continuous in $\mathbf{R}$ because it's a polynomial. $f(0) = 3$ and $f(-1) = 1 - 2 - 4 + 3 = -2$. Thus $y = 0$ is between $f(-1) = -2$ and $f(0) = 3$. By the Intermediate Value Theorem there is $z \in [-1, 0]$ such that $f(z) = 0$.

(b) STATEMENT: Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a continuous but unbounded function. Then f cannot be uniformly continuous.

FALSE: Let $f(x) = x$. f is unbounded, but it is uniformly continuous. Indeed, for all $\varepsilon > 0$ let $\delta = \varepsilon$. Then whenever $x, y \in \mathbf{R}$ satisfy $|x - y| < \delta$ we have $|f(x) - f(y)| = |x - y| < \delta = \varepsilon$.

(c) STATEMENT: Let $I = [-1, 1]$ and $f : I \rightarrow \mathbf{R}$ be a continuous function. If $\{x_n\} \subset I$ is a maximizing sequence ($\lim_{n \rightarrow \infty} f(x_n) = \sup_I f$) then $\{x_n\}$ converges in I .

FALSE: Let $f(x) = x^2$. Then f has two maxima at the endpoints of the interval. Then $x_n = (-1)^n$ is a maximizing sequence because $\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} 1 = 1 = \sup_I f$.

However, $x_n = (-1)^n$ does not converge.

4. Define: $\{S_n\}$ is a Cauchy Sequence. Consider the partial sums $S_n = \sum_{k=1}^n \frac{(-1)^k k}{4^k}$. Prove that the sequence $\{S_n\}$ is convergent.

$\{S_n\}$ is a Cauchy Sequence if for all $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|S_\ell - S_m| < \varepsilon \quad \text{whenever } \ell, m \in \mathbb{N} \text{ satisfy } \ell > N \text{ and } m > N.$$

To prove that $\{S_n\}$ is convergent, it suffices to show that it is a Cauchy Sequence. We shall use the fact that $k \leq 2^k$ for all $k \in \mathbb{N}$, which we proved when discussing induction. Choose $\varepsilon > 0$. Let $N = \frac{\ln(\frac{1}{\varepsilon})}{\ln 2}$. For any $\ell, m \in \mathbb{N}$ such that $\ell, m > N$ we either have $\ell = m$ in which case $|S_\ell - S_m| = 0 < \varepsilon$ or $\ell \neq m$. WLOG we may assume $\ell > m$ because otherwise we may swap the roles of ℓ and m . Then

$$\begin{aligned} |S_\ell - S_m| &= \left| \sum_{k=1}^{\ell} \frac{(-1)^k k}{4^k} - \sum_{k=1}^m \frac{(-1)^k k}{4^k} \right| = \left| \sum_{k=m+1}^{\ell} \frac{(-1)^k k}{4^k} \right| \leq \sum_{k=m+1}^{\ell} \frac{|(-1)^k k|}{4^k} \\ &= \sum_{k=m+1}^{\ell} \frac{k}{4^k} \leq \sum_{k=m+1}^{\ell} \frac{2^k}{4^k} = \sum_{k=m+1}^{\ell} \frac{1}{2^k} = \frac{1}{2^{m+1}} \sum_{k=0}^{\ell-m-1} \frac{1}{2^k} \\ &= \frac{1}{2^{m+1}} \cdot \frac{1 - (\frac{1}{2})^{\ell-m}}{1 - \frac{1}{2}} \leq \frac{1}{2^m} < \frac{1}{2^N} = \varepsilon. \end{aligned}$$

We have used that formula for a geometric sum. If $|r| < 1$ then

$$\sum_{k=0}^p r^k = \frac{1 - r^{p+1}}{1 - r}.$$

5. Let $\mathcal{D} \subset \mathbf{R}$ and $f : \mathcal{D} \rightarrow \mathbf{R}$. State the definition: f is uniformly continuous in $\mathcal{D}$. Let $f(x) = \frac{x^2}{1+x}$. Determine whether f is uniformly continuous on $\mathcal{D} = (0, \infty)$ and prove your answer using only your definition in (a).

f is a uniformly continuous on $\mathcal{D}$ if for all $\varepsilon > 0$ there is an $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever } x, y \in \mathcal{D} \text{ satisfy } |x - y| < \delta.$$

We show that $f(x) = \frac{x^2}{1+x}$ is uniformly continuous on $(0, \infty)$. To see it, choose $\varepsilon > 0$. Let $\delta = \varepsilon$. For any $x, y \in (0, \infty)$ such that $|x - y| < \delta$ we have

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{x^2}{1+x} - \frac{y^2}{1+y} \right| = \left| \frac{x^2(1+y) - y^2(1+x)}{(1+x)(1+y)} \right| = \frac{|x^2 - y^2 + x^2 y - xy^2|}{(1+x)(1+y)} \\ &= \frac{|(x+y+xy)(x-y)|}{(1+x)(1+y)} \leq \frac{1+x+y+xy}{(1+x)(1+y)} |x-y| = |x-y| < \delta = \varepsilon. \end{aligned}$$