Math 3210 § 2. **Third Midterm Exam** Name: Solutions
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1. *Let $\mathcal{D} \subset \mathbf{R}$, $a \in \mathcal{D}$ and $f : \mathcal{D} \to \mathbf{R}$. State the definition: $f(x)$ is* uniformly continuous *on $\mathcal{D}$. Using just your definition from prove that $f(x) = \dfrac{1}{\sqrt{x}}$ is uniformly continuous on (a, ∞), where $a > 0$.*

   Let $\mathcal{D} \subset \mathbf{R}$ and $f : \mathcal{D} \to \mathbf{R}$. Then f is *uniformly continuous* on $\mathcal{D}$ if for every $\varepsilon > 0$ there is an $\delta > 0$ such that

   $$|f(x) - f(y)| < \varepsilon \qquad \text{whenever } x, y \in \mathcal{D} \text{ are such that } x - y| < \delta.$$

   We show that $f(x)$ is Lipschitz on $\mathcal{D} = (a, \infty)$ and hence it is uniformly continuous. For any $x, y \in \mathcal{D}$ we have $a < x$ and $a < y$ so that

   $$\begin{aligned} |f(x) - f(y)| = \left| \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{y}} \right| &= \frac{|\sqrt{y} - \sqrt{x}|}{\sqrt{x}\,\sqrt{y}} = \frac{|\sqrt{y} - \sqrt{x}|}{\sqrt{x}\,\sqrt{y}} \cdot \frac{|\sqrt{y} + \sqrt{x}|}{|\sqrt{y} + \sqrt{x}|} \\ &= \frac{|y - x|}{\sqrt{x}\,\sqrt{y}\,|\sqrt{y} + \sqrt{x}|} \le \frac{|y - x|}{\sqrt{a}\,\sqrt{a}\,|\sqrt{a} + \sqrt{a}|} = \frac{1}{2a^{3/2}}|x - y|. \end{aligned}$$

   To see uniformly continuous, choose $\varepsilon > 0$. Let $\delta = 2a^{3/2}\varepsilon$. Then for any $x, y \in \mathcal{D}$ such that $|x - y| < \delta$ we have

   $$|f(x) - f(y)| \le \frac{1}{2a^{3/2}}|x - y| < \frac{\delta}{2a^{3/2}} = \frac{2a^{3/2}\varepsilon}{2a^{3/2}} = \varepsilon.$$

   There was some confusion: some students thought that the question was to show $f(x)$ is continuous at $a > 0$ in $(0, \infty)$.

2. *Let $I \subset \mathbf{R}$ be an open interval and $f : I \to \mathbf{R}$. State the definition: $f(x)$ is* differentiable *at $a \in I$. Suppose $f : \mathbf{R} \to \mathbf{R}$ is differentiable on $\mathbf{R}$ and that $f'(1) > 0$ and $f'(2) < 0$. Show there is $c \in (1, 2)$ such that $f'(c) = 0$.*

   Let $I \subset \mathbf{R}$ be an open interval and $f : I \to \mathbf{R}$. Then $f(x)$ is *differentiable* at $a \in I$ if the limit exists and is finite

   $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

   We say f is differentiable on I if it is differentiable at all points $a \in I$.

   This was one of your homework problems, called the intermediate value theorem for derivatives. The idea is to argue that because f is increasing at $x = 1$ and decreasing at $x = 2$, then in between f achieves its maximum over $[1, 2]$ at $1 < c < 2$ where $f'(c) = 0$.

   Because f is differentiable on $\mathbf{R}$ it is continuous on $\mathbf{R}$. Since a continuous function attains its maximum on a closed bounced interval, there is a $c \in [1, 2]$ such that

   $$f(c) = \sup_{x \in [1,2]} f(x).$$

   The maximizing point c can be neither $c = 1$ or $c = 2$. To rule out $c = 1$ consider the derivative

   $$0 < f'(1) = \lim_{y \to 1+} \frac{f(y) - f(1)}{y - 1}.$$

   Since the limit is positive, for y near 1 the difference quotient will also be positive: there is $\eta > 0$ so that

   $$0 < \frac{f(y) - f(1)}{y - 1} \qquad \text{whenever } 1 < y < 1 + \eta.$$

For such y we have $f(y) - f(1) > 0$ or $f(y) > f(1)$. Thus $x = 1$ is not the maximum of f on $[1, 2]$. Similarly, to rule out $c = 2$ consider the derivative

$$0 > f'(2) = \lim_{z \to 2-} \frac{f(z) - f(2)}{z - 2}.$$

Since the limit is negative, for z near 2 the difference quotient will also be negative: there is $\theta > 0$ so that

$$0 > \frac{f(z) - f(2)}{z - 2} \qquad \text{whenever } 2 - \theta < z < 2.$$

For such z we have $f(z) - f(1) > 0$ since $z - 2 < 0$. Thus $x = 2$ is not the maximum of f on $[1, 2]$.

Thus the maximum occurs at an interior point $c \in (1, 2)$. The maximum point is critical, and as f is differentiable at c we have $f'(c) = 0$.

The condition that f is differentiable on $\mathbf{R}$ says nothing about the continuity of f'. Indeed, there is an example of Weierstrass of a continuous function on $\mathbf{R}$ which is nowhere differentiable. If we knew that $f'(x)$ were continuous, then the statement would follow from the ordinary Intermediate Value Theorem applied to $f'(x)$ on $[1, 2]$.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT: *Suppose that $f : [0, 1] \to \mathbf{R}$ is continuous and strictly invreasing on $[0, 1]$. If f is differentiable at $a \in (0, 1)$ then the inverse function is differentiable at $b = f(a)$.*

   FALSE: The statement would be true if we knew that $f'(a) \neq 0$. For example, $f(x) = (x - \frac{1}{2})^3$ is strictly increasing and differentiable, but the inverse function $g(y) = y^{1/3} + \frac{1}{2}$ is not differentiable at $0 = f(\frac{1}{2})$.

   (b) STATEMENT: *If $f, g : [0, 1] \to \mathbf{R}$ are differentiable, $f(0) = g(0)$ and $f'(x) \le g'(x)$ for all x, then $f(x) \le g(x)$ for $0 < x \le 1$.*

   TRUE: This follows from the Mean Value Theorem. Let $h(x) = g(x) - f(x)$. Then h is continuous on $[0, x]$ and differentiable on $(0, x)$ for $0 < x \le 1$. By the Mean Value Theorem, there is $c \in (0, x)$ so that

   $$g(x) - f(x) - 0 = h(x) - h(0) = h'(c)(x - 0) \ge 0$$

   since $h'(c) = g'(c) - f'(c) \ge 0$. Thus $g(x) \ge f(x)$.

   (c) STATEMENT: *Suppose $f : [0, 1] \to [0, 1]$ is integrable. Then there is a $c \in [0, 1]$ where $f(c) = c$.*

   FALSE: The statement would have been true if we knew that $f(x)$ were continuous using the Intermediate Value Theorem. However, integrable functions need not be continuous. For example

   $$f(x) = \begin{cases} 1, & \text{if } x \le \frac{1}{2}; \\ 0, & \text{if } x > \frac{1}{2}. \end{cases}$$

   f is continuous except at one point, thus is integrable. But $f(x) \neq x$ for all x.

4. *State the definitions: Let $f, f_n : \mathbf{R} \to \mathbf{R}$ be functions. (i)* $\{f_n(x)\}$ converges pointwise *to a function f on $\mathbf{R}$ as $n \to \infty$. (ii)* $\{f_n(x)\}$ converges uniformly *to a function f on $\mathbf{R}$ as $n \to \infty$.*

   Determine whether the functions $f_n(x) = \dfrac{(x+n)^2}{1 + (x+n)^2}$ *converge pointwise, converge uniformly, or do not converge to a function $f(x)$ on $\mathbf{R}$ and prove your result.*

Definitions: (i) $\{f_n(x)\}$ *converges pointwise* to a function f on $\mathbf{R}$ as $n \to \infty$ if for all $x \in \mathbf{R}$ and all $\varepsilon > 0$ there is an $N \in \mathbf{R}$ so that

$$|f_n(x) - f(x)| < \varepsilon \qquad \text{whenever } n > N.$$

(ii) $\{f_n(x)\}$ *converges uniformly* to a function f on $\mathbf{R}$ as $n \to \infty$ if for all $\varepsilon > 0$ there is an $N \in \mathbf{R}$ so that

$$|f_n(x) - f(x)| < \varepsilon \qquad \text{whenever } n > N \text{ and any } x \in \mathbf{R}.$$

The functions $f_n(x) = \dfrac{(x+n)^2}{1+(x+n)^2}$ converge uniformly to a function $f(x)$ on $\mathcal{D} = (0, \infty)$. Determine the pointwise limit using limit rules. For $x \in \mathcal{D}$,

$$f(x) = \lim_{n\to\infty} \frac{(x+n)^2}{1+(x+n)^2} = \lim_{n\to\infty} \frac{1}{\frac{1}{(x+n)^2} + 1} = \frac{1}{0+1} = 1.$$

To see the convergence is uniform, choose $\varepsilon > 0$. Let $N = 1/\sqrt{\varepsilon}$. Then for any $x \in \mathcal{D}$ we have for $n > N$ and using $x > 0$,

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \frac{(x+n)^2}{1+(x+n)^2} - 1 \right| = \left| \frac{(x+n)^2}{1+(x+n)^2} - \frac{1+(x+n)^2}{1+(x+n)^2} \right| \\ &= \frac{1}{1+(x+n)^2} < \frac{1}{1+n^2} < \frac{1}{n^2} < \frac{1}{N^2} = \varepsilon. \end{aligned}$$

5. *Let f be a bounded function on the closed bounded interval $[a,b]$. Complete the statement of the theorem. [Of several possible answers, select the one you prefer for the next question.]*

Theorem. The bounded function f on the closed bounded interval $[a,b]$ is integrable if and only if

Using only the theorem above, show that if $f(x)$ is integrable on $[a,b]$ then the positive part f^+ is integrable on $[a,b]$, where $f^+(x) = \begin{cases} f(x), & \text{if } f(x) > 0; \\ 0, & \text{if } f(x) \le 0. \end{cases}$

Theorem. The bounded function f on the closed bounded interval $[a,b]$ is integrable if and only if for every $\varepsilon > 0$ there is a partition $\mathcal{P}$ of $[a,b]$ such that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$.

We show f^+ is integrable. Choose $\varepsilon > 0$. By the Theorem, there is a partition $\mathcal{P}$ such that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$. To show this for f^+, let the partition be $\mathcal{P} = \{a = x_0 \le x_1 \le \cdots \le x_n = b\}$ and for each subinterval $I_k = [x_{k-1}, x_k]$ we define for a bounded function $g : [a,b] \to \mathbf{R}$

$$m_i(g) = \inf_{I_k} g, \qquad M_i(g) = \sup_{I_k} g.$$

Claim: $M_k(f^+) - m_k(f^+) \le M_k(f) - m_k(f)$ for all k. The simplest argument for the claim is to consider three cases. In case $0 \le m_k(f) \le M_k(f)$ then $f \ge 0$ in I_k and $f^+ = f$ over I_k. In this case $M_k(f^+) - m_k(f^+) = M_k(f) - m_k(f)$. In case $m_k(f) \le 0 \le M_k(f)$ then $m_k(f^+) = 0$

and $M_k(f^+) = M_k(f)$. In this case $M_k(f^+) - m_k(f^+) = M_k(f) - 0 \le M_k(f) - m_k(f)$. In the last case $m_k(f) \le M_k(f) \le 0$. Then $f \le 0$ in I_k so $f^+ = 0$ over I_k. In this case $M_k(f^+) - m_k(f^+) = 0 - 0 \le M_k(f) - m_k(f)$. Thus in all three cases the claim has been verified. Adding, we see from the claim that

$$\begin{aligned} U(f^+, \mathcal{P}) - L(f^+, \mathcal{P}) &= \sum_{k=1}^{n} (M_k(f^+) - m_k(f^+))(x_k - x_{k-1}) \\ &\le \sum_{k=1}^{n} (M_k(f) - m_k(f))(x_k - x_{k-1}) = U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon. \end{aligned}$$

Thus, by the Theorem, f^+ is integrable on $[a, b]$.