Math 3210 § 2. **Second Midterm Exam** Name: Solutions
Treibergs February 25, 2026.

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1. *Let* $a_n = \dfrac{n}{\sqrt{n^2+1}}$. *Define:* $L = \lim_{n\to\infty} a_n$. *Find* L *using limit laws.*
Prove using just your definition that $L = \lim_{n\to\infty} a_n$.

The number $L \in \mathbf{R}$ is the limit of the sequence $L = \lim_{n\to\infty} a_n$ if for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|a_n - L| < \varepsilon \qquad \text{whenever } n > N.$$

Computing using limit laws we find

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty}\sqrt{1+\frac{1}{n^2}}} = \frac{1}{\sqrt{\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)}} \\ &= \frac{1}{\sqrt{\lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{1}{n^2}}} = \frac{1}{\sqrt{1+\left(\lim_{n\to\infty}\frac{1}{n}\right)^2}} = \frac{1}{\sqrt{1+0^2}} = 1. \end{aligned}$$

To prove $L = 1$, choose $\varepsilon > 0$. Let $N = \dfrac{1}{\sqrt{2\varepsilon}}$. Then for any $n \in \mathbb{N}$ such that $n > N$ we have

$$\begin{aligned} |a_n - L| &= \left|\frac{n}{\sqrt{n^2+1}} - 1\right| = \left|\frac{n-\sqrt{n^2+1}}{\sqrt{n^2+1}}\right| = \left|\frac{n-\sqrt{n^2+1}}{\sqrt{n^2+1}}\cdot\frac{n+\sqrt{n^2+1}}{n+\sqrt{n^2+1}}\right| \\ &= \left|\frac{n^2-(n^2+1)}{\sqrt{n^2+1}(n+\sqrt{n^2+1})}\right| = \frac{1}{\sqrt{n^2+1}(n+\sqrt{n^2+1})} < \frac{1}{2n^2} < \frac{1}{2N^2} = \varepsilon. \end{aligned}$$

2. *Let* $\mathcal{D} \subset \mathbf{R}$ *be a nonempty set,* $a \in \mathcal{D}$ *and* $f : \mathcal{D} \to \mathbf{R}$ *be a function. Define:* f is continuous at a in $\mathcal{D}$. *State a Theorem that characterizes continuity of a function at a in* $\mathcal{D}$ *in terms of sequences. Let* $f(x) = x^3 + x^2$. *Give two proofs for the continuity of* $f(x)$ *at* a *in* $\mathbf{R}$, *one using the only the definition and the other using the sequential characterization.*

Let $\mathcal{D} \subset \mathbf{R}$ be a nonempty set, $a \in \mathcal{D}$ and $f : \mathcal{D} \to \mathbf{R}$ be a function. f is *continuous at a in* $\mathcal{D}$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \qquad \text{whenever } x \in \mathcal{D} \text{ such that } |x - a| < \delta.$$

Theorem 1 [Sequential Characterization of Continuity] *Let* $\mathcal{D} \subset \mathbf{R}$ *be a nonempty set,* $a \in \mathcal{D}$ *and* $f : \mathcal{D} \to \mathbf{R}$ *be a function.* f *is continuous at* a *in* $\mathcal{D}$ *if and only if for every sequence* $\{x_n\} \subset \mathcal{D}$ *such that* $x_n \to a$ *as* $n \to \infty$ *there holds* $f(x_n) \to f(a)$ *as* $n \to \infty$.

Proof of continuity of $f(x) = x^3 + x^2$ at a in $\mathbf{R}$ using the definition: Choose $\varepsilon > 0$. Let

$$\delta = \min\left\{1, \frac{\varepsilon}{3|a|^2 + 5|a| + 2}\right\}.$$

For any $x \in \mathbf{R}$ such that $|x - a| < \delta$, since $\delta \le 1$ we have using the triangle inequality

$$|x| = |a - (a - x)| \le |a| + |a - x| \le |a| + 1.$$

Furthermore, by the triangle inequality

$$\begin{aligned}|f(x) = f(a)| &= |x^3 + x^2 - (a^3 + a^2)| \\ &= |x^3 - a^3 + x^2 - a^2| \\ &= |(x^2 + ax + a^2)(x - a) + (x + a)(x - a)| \\ &= |x^2 + ax + a^2 + x + a|\,|x - a| \\ &\le \big((|a| + 1)^2 + |a|(|a| + 1) + |a|^2 + |a| + 1 + |a|\big)|x - a| \\ &\le (3|a|^2 + 5|a| + 2)|x - a| \\ &< (3|a|^2 + 5|a| + 2) \cdot \frac{\varepsilon}{3|a|^2 + 5|a| + 2} = \varepsilon,\end{aligned}$$

completing the proof.

Proof of continuity of $f(x) = x^3+x^2$ at a in $\mathbf{R}$ using the sequential characterization: Choose a sequence $\{x_n\} \subset \mathbf{R}$ such that $x_n \to a$ as $n \to \infty$. By the Workhorse Theorem, the limit of powers and sums is the power and sum of limits, resp. Thus

$$f(x_n) = x_n^3 + x_n^2 \to a^3 + a^2 = f(a) \qquad \text{as } n \to \infty.$$

It follows from the Sequential Characterization that $f(x)$ is continuous at a in $\mathbf{R}$.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) Statement: *Define recursively* $x_1 = 0$ *and* $x_{n+1} = f(x_n) = \frac{1}{2} + \sin(x_n)$ *for all* $n \ge 1$. *Then the sequence* $\{x_n\}$ *converges to a real number.*

   True. The sequence is strictly increasing and bounded above. Hence, by the Monotone Convergence Theorem, there is $L \in \mathbf{R}$ such that $x_n \to L$ as $n \to \infty$.

   To see that the sequence is bounded and increasing, that is $x_n \le \frac{3}{2}$ and $x_{n+1} > x_n$ for every n in $\mathbf{N}$, we argue by induction. For the base case $n = 1$ we have given $x_1 = 0 \le \frac{3}{2}$ and $x_2 = \frac{1}{2} + \sin(x_1) = \frac{1}{2} + 0 = \frac{1}{2} > 0 = x_1$.

   For the induction case, assume that for some $n \in \mathbb{N}$ we have $x_n \le \frac{3}{2}$ and $x_{n+1} > x_n$. Then $x_{n+1} = \frac{1}{2} + \sin(x_n) \le \frac{1}{2} + 1 = \frac{3}{2}$. Also, since $f(x)$ is strictly increasing on $[0, \frac{3}{2}] \subset [0, \frac{\pi}{2}]$, because of the induction hypothesis, $x_{n+1} > x_n$ which are both in $[0, \frac{3}{2}]$, we have $x_{n+2} = f(x_{n+1}) > f(x_n) = x_{n+1}$. Thus the statement holds for $n + 1$. By mathematical induction, $x_n \le \frac{3}{2}$ and $x_{n+1} > x_n$ for all $n \in \mathbf{N}$.

   In fact, by the spiderweb diagram for this iteration sequence, the terms increase toward the intersection of the lines $y = \frac{1}{2} + \sin x$ and $y = x$ which occurs about $x = L = 1.493003890\ldots$.

   (b) Statement: *If the real sequence* $\{x_n\}$ *has a convergent subsequence then the whole sequence* $\{x_n\}$ *converges too.*

   False. Consider the sequence $x_n = (-1)^n$. It has a convergent subsequence $x_{2n} = 1$ but does not converge itself.

   (c) Statement: *Let* I_n *be nonempty bounded intervals such that* $I_1 \supset I_2 \supset I_2 \supset \cdots$. *Then* $\bigcap_{k=1}^{\infty} I_n \ne \emptyset$.

   False. The sets need not be bounded. Thus if we take $I_n = [n, \infty)$, this forms a decreasing sequence of closed sets such that $\bigcap_{k=1}^{\infty} I_n = \emptyset$.

4. *Let* $I = [a, b]$ *be a closed bounded interval and* $f : I \to \mathbf{R}$ *a continuous function. Assuming that* f *is bounded below on* I, *show that there is a point* $c \in I$ *such that* $f(c) = \inf_I f$.

The question asks for the proof of the assertion; there is no point in just giving a restatement. We use the method of minimizing sequences.

Because the function is bounded below, there is a finite infimum $M = \inf\limits_{x \in I} f$. This means that M is a lower bound: $M \le f(x)$ for all $x \in I$ and that larger numbers are not lower bounds: for every $\varepsilon > 0$ there is an $x \in I$ such that $f(x) < M + \varepsilon$. Hence, for every $n \in \mathbb{N}$ there is $x_n \in I$ such that

$$M \le f(x_n) < M + \frac{1}{n}.$$

Thus by the Squeeze Theorem,

$$\lim_{n \to \infty} f(x_n) = M. \tag{1}$$

Because $\{x_n\} \subset I$, a bounded set, by the Bolzano-Weierstrass Theorem, there is a subsequence convergng to a real number $x_{n_k} \to c$ as $k \to \infty$. Because I is closed, $a \le x_{n_k} \le b$ for all k implies the limit $a \le c \le b$ or $c \in I$. Finally we claim $f(c) = M$, the desired point. By the sequential characterization of continuity of f at c in I, the fact that a subsequence of a convergent sequence takes the same limit and (1) we have that

$$f(c) = \lim_{k \to \infty} f(x_{n_k}) = \lim_{n \to \infty} f(x_n) = M.$$

5. *Let* $\{S_n\}$ *be a real sequence. State the definition:* $\{S_n\}$ is a Cauchy Sequence. *Let* $\{a_n\}$ *be a bounded real sequence and* $\{S_n\}$ *be the partial sums* $S_n = \sum\limits_{k=1}^{n} \dfrac{a_k}{2^k}$. *Show that* $\{S_n\}$ *converges to a real number* $S = \lim\limits_{n \to \infty} S_n$.

The real sequence $\{S_n\}$ is a *Cauchy Sequence* if for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|S_i - S_j| < \varepsilon \qquad \text{whenever } i > N \text{ and } j > n.$$

Because $\{a_n\}$ is a bounded sequence, there is $B \in \mathbf{R}$ so that

$$|a_n| < B \qquad \text{for all } n \in \mathbb{N}.$$

To show $S = \lim\limits_{n \to \infty} S_n$ we show that $\{S_n\}$ is a Cauchy Sequence, hence convergent. To see that $\{S_n\}$ is a Cauchy Sequence, choose $\varepsilon > 0$. Let $N = \dfrac{\ln B/\varepsilon}{\ln 2}$ so that $\dfrac{B}{2^N} = \varepsilon$. For any $i, j \in \mathbb{N}$ such that $i > N$ and $j > N$ we may assume without loss of generality that $i > j$. Indeed, if $i = j$ we have $|S_i - S_j| = 0 < \varepsilon$ and if $i < j$, use $|S_i - S_j| = |S_j - S_i|$ and swap the roles of i and j. Then using the triangle inequality, the bound on $\{a_k\}$ and the sum of a geometric series

$$\begin{aligned} |S_i - S_j| &= \left| \sum_{k=1}^{i} \frac{a_k}{2^k} - \sum_{k=1}^{j} \frac{a_k}{2^k} \right| = \left| \sum_{k=j+1}^{i} \frac{a_k}{2^k} \right| \le \sum_{k=j+1}^{i} \frac{|a_k|}{2^k} \le \sum_{k=j+1}^{i} \frac{B}{2^k} \\ &= \frac{B}{2^{j+1}} \sum_{k=0}^{i-j-1} \frac{1}{2^k} = \frac{B}{2^{j+1}} \cdot \frac{1 - \left(\frac{1}{2}\right)^{i-j}}{1 - \frac{1}{2}} < \frac{B}{2^j} < \frac{B}{2^N} = \varepsilon. \end{aligned}$$