Math 3210 § 1. **First Midterm Exam** Name: Solutions
Treibergs September 17, 2008

(1.) *Prove that for all* $n \in \mathbb{N}$, $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

Proof by induction. In the base case, $n = 1$, the left side is $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} = \frac{1}{2}$. The right side is $\frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}$ hence equality holds.

The induction setep is to prove the statement for $n+1$ assuming it's true for n. But

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{1}{(n+1)(n+2)} + \sum_{i=1}^{n} \frac{1}{i(i+1)} \qquad \text{Now use the induction hypothesis.}$$

$$= \frac{1}{(n+1)(n+2)} + \frac{n}{n+1} = \frac{1+n(n+2)}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{(n+1)+1}.$$

(2.) *Define a new binary operator* $\boxplus$ *on* $\mathbb{N}$ *as follows. For each* $m \in \mathbb{N}$, *the operation is defined recursively: let* $1 \boxplus m := m + 1$ *and for* $n \ge 1$, *let* $(n+1) \boxplus m := (n \boxplus m) + 1$. *How is ordinary addition "+" defined on* $\mathbb{N}$? *Show that for all* $m, n \in \mathbb{N}$, $n \boxplus m = m + n$, *where "+" is ordinary addition.*

Usual addition "+" is also defined recursively. Let $m \in \mathbb{N}$. Then $m+1 := m+1$, the successor to m, and for $n \ge 1$, $m + (n+1) := (m+n) + 1$.

Let's prove the statement $\mathcal{P}(n) \iff$ "$n \boxplus m = m + n$" using induction. The base case $\mathcal{P}(1)$, $1 \boxplus m = m + 1$ is the base case for the definition of "$\boxplus$" and $m + 1 := m + 1$ is the base case for the definition of addition. Since they are equal, $\mathcal{P}(1)$ holds.

The induction step is to show $\mathcal{P}(n+1)$ assuming $\mathcal{P}(n)$ for $n \ge 1$. But $(n+1) \boxplus m = (n \boxplus m) + 1$ by the inductive definition of "$\boxplus$." By the induction hypothesis, $\mathcal{P}(n)$, this equals $(m+n)+1$. By the inductive definition of addition (or by associativity of addition) this equals $m + (n+1)$. Thus we have shown $\mathcal{P}(n+1)$.

(3.) *Let* $f : X \to Y$ *be a function. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

Statement A. *If* $X = f^{-1}(Y)$ *then* f *is onto.*

FALSE. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$ then $f^{-1}(\mathbb{R}) = \mathbb{R}$ but f is not onto since $-4 \in \mathbb{R}$ is not in the image since $f(x) \ge 0$ for all $x \in \mathbb{R}$. In fact, $X = f^{-1}(Y)$ is true for every function.

Statement B. *Suppose that for all* $x_1, x_2 \in X$ *such that* $f(x_1) \ne f(x_2)$ *we have* $x_1 \ne x_2$. *Then* f *is one-to-one.*

FALSE. Same example as in A. The logically equivalent contrapositive statement is $x_1 = x_2$ implies $f(x_1) = f(x_2)$ which is true for every function, not just one-to-one functions. Thus for $f(x) = x^2$ we have $x_1^2 \ne x_2^2$ implies $x_1 \ne x_2$ but f is not one-to-one since $f(-2) = 4 = f(2)$.

Statement C. *If* $A \subset X$ *and* $f(A) = Y$ *then* $A = X$.

FALSE. Define $f : \mathbb{R} \to [0, \infty)$ by $f(x) = x^2$. Let $X = \mathbb{R}$ and $A = Y = [0, \infty)$. Then $f(A) = Y$ but $A \ne X$.

(4.) *Show that if* a *and* b *are elements of the commutative ring* $(R, +, \cdot)$, *then* $x = (-a) + b$ *solves the equation* $a + x = b$. *Show that the solution is unique.*

$$\begin{aligned} a + x &= a + ((-a) + b) \\ &= (a + (-a)) + b && \text{by associativitiy of addition A2;} \\ &= 0 + b && \text{by property of additive inverse A4;} \\ &= b && \text{by property of additive identity, A3.} \end{aligned}$$

Thus x solves the equation $a + x = b$. Suppose y is another solution. Then

$$\begin{aligned} a + y &= b \\ (-a) + (a + y) &= (-a) + b, && \text{By adding } -a \text{ to both sides.} \\ ((-a) + a) + y &= (-a) + b, && \text{by associativity of addition A2;} \\ (a + (-a)) + y &= (-a) + b, && \text{by commutitivity of addition A1;} \\ 0 + y &= (-a) + b, && \text{by property of additive inverse A4;} \\ y &= (-a) + b, && \text{by property of additive identity, A3.} \end{aligned}$$

Thus another solution must equal the first, so the solution is unique.

(5.) *Give as simple a description as possible of the set $\mathcal{S}$ in terms of intervals of the real numbers*

$$\mathcal{S} = \{x \in \mathbb{R} : (\exists m \in \mathbb{N})(\forall \epsilon \in \mathbb{R} \text{ such that } \epsilon > 0) \quad m - \epsilon < x\}.$$

Show that set you describe equals $\mathcal{S}$.

$$\mathcal{S} = \left\{ x \in \mathbb{R} : (\exists m \in \mathbb{N}) \left(x \in \bigcap_{\epsilon > 0} (m - \epsilon, \infty) \right) \right\} = \bigcup_{m \in \mathbb{N}} \bigcap_{\epsilon > 0} (m - \epsilon, \infty) = \bigcup_{m \in \mathbb{N}} [m, \infty) = [1, \infty).$$

To show $\mathcal{S} = [0, \infty)$ we prove "$\subset$" and "$\supset$."

To show "$\subset$", we choose any $y \in \mathcal{S}$ to show $y \in [1, \infty)$. But $y \in \mathcal{S}$ means for some $m_0 \in \mathbb{N}$ we have for all $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$ there holds $m_0 - \epsilon < y$. Hence, $y \geq m_0$ since otherwise, $y < m_0$ implies that for some $\epsilon_0 \in \mathbb{R}$ such that $0 < \epsilon_0 < m_0 - y$ we have $m_0 - \epsilon_0 > y$ contrary to $m - \epsilon < y$ for all $\epsilon > 0$. Finally, since $y \geq m_0 \geq 1$ we have $y \in [1, \infty)$.

To show "$\supset$," we choose $y \in [1, \infty)$ to show $y \in \mathcal{S}$. Then for $m_0 = 1$ we have $y \geq m_0$. Thus, for every $\epsilon > 0$ we have $y > m_0 - \epsilon$. In other words $(\forall \epsilon \in \mathbb{R} : \epsilon > 0)(m_0 - \epsilon < y)$. Since $m_0 \in \mathbb{N}$ we also have

$$(\exists m \in \mathbb{N})(\forall \epsilon \in \mathbb{R} : \epsilon > 0)(m - \epsilon < y).$$

Thus y satisfies the condition to belong to $\mathcal{S}$.