Math 3210 § 1. **Second Midterm Exam** Name: Solutions
Treibergs October 8, 2008

1. *Let L be a real number and $\{a_n\}_{n\in\mathbb{N}}$ be a real sequence. State the definition: $L = \lim_{n\to\infty} a_n$. Let $a_n = \frac{n}{3n-8}$. Find $\lim_{n\to\infty} a_n$. Prove your answer is the limit.*

Definition of $\lim_{n\to\infty} a_n = L$: for every $\epsilon > 0$ there is $R \in \mathbb{R}$ so that $|a_n - L| < \epsilon$ whenever $n > R$.

To show $a_n \to \frac{1}{3}$ as $n \to \infty$. Choose $\epsilon > 0$. Let $R = \frac{4}{3\epsilon} + 8$. If $n > R$ then $n > 8$, $3n - 8 > 0$ and $\frac{1}{R} < \frac{3\epsilon}{4}$ so

$$|a_n - L| = \left|\frac{n}{3n-8} - \frac{1}{3}\right| = \left|\frac{3n-(3n-8)}{3(3n-8)}\right| = \frac{8}{3(3n-8)} \le \frac{8}{3(3n-n)} = \frac{4}{3n} < \frac{4}{3R} < \epsilon. \quad \square$$

2. *Let $(\mathcal{F}, +, \cdot, \le)$ be an ordered field. For $x \in \mathcal{F}$, what does $x > 0$ mean? For $x, y \in \mathcal{F}$ show that $x > 0$ and $y > 0$ implies $x + y > 0$.*

$x > 0$ means $x \ge 0$ and $x \ne 0$. We need to show that $x + y \ge 0$ and $x + y \ne 0$. Since $x > 0$ and $y > 0$ we have $x \ge 0$ and $y \ge 0$. By **O4**(if $p \le q$ then $p + r \le q + r$), adding y to both sides, $x + y \ge 0 + y = y \ge 0$ by additive identity and $y \ge 0$. Thus we have shown $x + y \ge 0$. To show that $x + y \ne 0$ assume the contrary, that $x + y = 0$. But since the additive inverse is unique in a field, $y = -x$ so $-x = y > 0$. But adding $-x$ to $x \ge 0$ yields $0 = x - x \ge 0 - x = -x$. But $-x > 0$ and $-x \le 0$ implies $-x \ne 0$ and $-x \ge 0$ and $-x \le 0$ implies $-x \ne 0$ and $-x = 0$ by **O2**(if $y \le z$ and $z \le y$ then $y = z$), which is a contradiction.

3a. Statement: *If $a, b \in \mathbb{R}$ then $|a + b| \ge |a| - |b|$.*

TRUE. By the triangle inequality $|a| = |-b + (a + b)| \le |-b| + |a + b| = |b| + |a + b|$.

3b. Statement: *Suppose that $A \subset \mathbb{R}$ is a nonempty subset such that $\sup A = \infty$. Then for every $b \in \mathbb{R}$ there are only finitely many $a \in A$ so that $a \le b$.*

FALSE. Let $A = \mathbb{N} \cup \{\frac{1}{m} : m \in \mathbb{N}\}$. Then $\sup A = \infty$ because A is not bounded above: for every $b \in \mathbb{R}$ there is $n \in \mathbb{N} \subset A$ such that $n > b$ by the Archimedian property. However, for $b = \frac{3}{2}$, every one of the infinitely many numbers $\frac{1}{m} \in A$ for $m \in \mathbb{N}$ satisfy $\frac{1}{m} < b$.

3c. Statement: *For every $\epsilon > 0$ there is a rational number r such that $|r - \sqrt{2}| < \epsilon$.*

TRUE. For every $\epsilon > 0$, by the density of rationals, there is a rational number r between the real numbers $\sqrt{2} - \epsilon < r < \sqrt{2} + \epsilon$.

4. *Let L be a real number and $A \subset \mathbb{R}$ a nonempty subset. State the definition: $L = \sup A$. Let $A = \left\{\frac{n^2}{(n+1)^2} : n \in \mathbb{N}\right\}$. Find $\sup A$. Prove your answer is the supremum.*

Definition of $L = \sup A$: L is an upper bound for A, *i.e.,* $(\forall a \in A)(a \le L)$ and L is the least among upper bounds, *i.e.,* $(\forall \epsilon > 0)(\exists a \in A)(L - \epsilon < a)$.

To show $L = 1$ is the supremum of A we need to show 1 is an upper bound and that 1 is the least among upper bounds. But if $n \in \mathbb{N}$ we have $0 < n < n + 1$ which implies $0 < n^2 < n(n+1) < (n+1)^2$ so that $a = \frac{n^2}{(n+1)^2} < 1$. Since every $a \in A$ has this form, $(\forall a \in A)(a \le 1)$ so 1 is an upper bound.

To show it is least among upper bounds, we choose $\epsilon > 0$ to show that there is $a \in A$ such that $1 - \epsilon < a$. By the Archimedian Property, there is $n \in \mathbb{N}$ so that $n > \frac{2}{\epsilon}$. I claim $a = \frac{n^2}{(n+1)^2} > 1 - \epsilon$. To see this,

$$1 - a = 1 - \frac{n^2}{(n+1)^2} = \frac{(n+1)^2 - n^2}{(n+1)^2} = \frac{2n+1}{(n+1)^2} < \frac{2n+2}{(n+1)^2} = \frac{2}{n+1} < \frac{2}{n} < \epsilon. \quad \square$$

5. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a real sequence and $c, d, L \in \mathbb{R}$ be numbers. Let $b_n = ca_n + d$. Show that if $a_n \to L$ as $n \to \infty$ then $b_n \to cL + d$ as $n \to \infty$.*

To show for all $\epsilon > 0$ there is $R \in \mathbb{R}$ so that $|b_n - cL - d| < \epsilon$ whenever $n > R$. Choose $\epsilon > 0$. By the assumption that $a_n \to L$ as $n \to \infty$, there is $R \in \mathbb{R}$ so that $n > R$ implies $|a_n - L| < \frac{\epsilon}{1+|c|}$. For the same R, suppose $n > R$. Then

$$|b_n - cL - d| = |ca_n + d - cL - d| = |c|\,|a_n - L| \le \frac{|c|\,\epsilon}{|c| + 1} < \epsilon. \quad \square$$