Math 3210 § 1. **Third Midterm Exam** Name: Solutions

Treibergs November 12, 2008

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[1] *Let $\{x_n\} \subset \mathbb{R}$ be a sequence. State the definition: $\{x_n\}$ is a Cauchy sequence. Let $x_n = \dfrac{n}{n+1}$. Show that $\{x_n\}$ is a Cauchy sequence.*

Definition. $\{x_n\}$ is a *Cauchy sequence* if for every $\epsilon > 0$ there is an $R \in \mathbb{R}$ such that $|x_n - x_m| < \epsilon$ whenever $m, n \in \mathbb{N}$ satisfy $m, n > R$.

Proof. Choose $\epsilon > 0$. Let $R = \frac{1}{\epsilon}$. Suppose $m, n \in \mathbb{N}$ satisfy $m, n > R$. One is larger, say, $m \geq n$. Then since $0 \leq m - n \leq m + 1$,

$$|x_n - x_m| = \left|\frac{m}{m+1} - \frac{n}{n+1}\right| = \frac{|m(n+1) - n(m+1)|}{(m+1)(n+1)} = \frac{m-n}{m+1} \cdot \frac{1}{n+1} \leq \frac{1}{n+1} < \frac{1}{R} = \epsilon. \quad \square$$

[2.] *Let $f, f_n : D \to \mathbb{R}$ be functions. State the definition: f_n converges to f pointwise on D as $n \to \infty$. State the definition: f_n converges to f uniformly on D as $n \to \infty$. Let $g_n(x) = \dfrac{x^2}{n^2 + x^2}$ and $g(x) = 0$. Show that $g_n \to g$ pointwise on $\mathbb{R}$. Does $g_n \to g$ uniformly on $\mathbb{R}$? Prove your answer.*

Definition: $f_n \to f$ *converges pointwise* on D means for every $x \in D$ we have $\lim_{n\to\infty} f_n(x) = f(x)$. *i.e.*, for every $x \in D$ and for every $\epsilon > 0$ there is $R \in \mathbb{R}$ such that $|f_n(x) - f(x)| < \epsilon$ whenever $n \in \mathbb{N}$ satisfies $n > R$.

Definition: $f_n \to f$ *converges uniformly* on D means for every $\epsilon > 0$ there is $R \in \mathbb{R}$ such that $|f_n(x) - f(x)| < \epsilon$ whenever $x \in D$ and $n \in \mathbb{N}$ satisfies $n > R$.

To see that $g_n \to g$ pointwise, by the workhorse theorem for sequences, for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{x^2}{n^2 + x^2} = \lim_{n\to\infty} \frac{\frac{x^2}{n^2}}{1 + \frac{x^2}{n^2}} = \frac{0}{1+0} = 0.$$

However, the convergence is not uniform. Negating the definition, g_n does not converge uniformly to g means there is an $\epsilon_0 > 0$ such that for every $R > 0$ there is an $n \in \mathbb{N}$ such that $n > R$ and there is an $x \in \mathbb{R}$ such that $|g_n(x) - g(g)| \geq \epsilon_0$. Take $\epsilon_0 = \frac{1}{2}$. Choose $R \in \mathbb{R}$. Take $n \in \mathbb{N}$ to satisfy $n > R$ (by the Archimedian property) and let $x = n$. Then

$$|g_n(x) - g(x)| = |g_n(n) - 0| = \left|\frac{n^2}{n^2 + n^2}\right| = \frac{1}{2} \geq \epsilon_0. \quad \square$$

[3.] *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

(a.) Statement. *Let $\{y_n\}$ be a sequence such that $y_n > 0$ for all n. If $y_n \to y$ as $n \to \infty$ then $y > 0$.*

FALSE. Let $y_n = \frac{1}{n}$. Then $y_n \to 0$ as $n \to \infty$ but 0 is not positive.

(b.) Statement. *Let $\{z_n\}$ be a sequence that has a convergent subsequence. Then $\{z_n\}$ is bounded.*

FALSE. Let $z_n = \begin{cases} 0, & \text{if } n \text{ is even;} \\ n, & \text{if } n \text{ is odd.} \end{cases}$. Then the even subsequence $z_{2n} = 0 \to 0$ but z_n is unbounded since $z_{2n+1} = 2n + 1 \to \infty$ as $n \to \infty$.

(c.) Statement. *If $f : (0, 1) \to \mathbb{R}$ is uniformly continuous then $\displaystyle\lim_{n\to\infty} f\left(\frac{1}{n}\right)$ exists.*

TRUE. A uniformly continuous function on a bounded open interval has a continuous extension on the closure, $F : [0, 1] \to \mathbb{R}$ such that $F = f$ on $(0, 1)$. But the sequence $\{\frac{1}{n}\} \subset [0, 1]$ tends to zero $\frac{1}{n} \to 0$ as $n \to \infty$, and since F is continuous at zero, $F(\frac{1}{n}) \to F(0)$ as $n \to \infty$.

[4.] *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that for some $x_0 \in \mathbb{R}$ we have $f(x_0) > 0$. Show that there are $a, b, c \in \mathbb{R}$ such that $a < b$ and $0 < c$ such that $f(x) \geq c$ whenever $a < x < b$.*

Proof. Since f is continuous at x_0, for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. Apply this to the special case $\epsilon_0 = \frac{1}{2} f(x_0) > 0$ and let $\delta_0 > 0$ be the corresponding δ. Then for $x \in (x_0 - \delta_0, x_0 + \delta_0)$ we have $|f(x) - f(x_0)| < \frac{1}{2} f(x_0)$. This implies for such x,

$$f(x) = f(x_0) + f(x) - f(x_0) \geq f(x_0) - |f(x) - f(x_0)| > f(x_0) - \frac{1}{2} f(x_0) = \frac{1}{2} f(x_0).$$

Thus we have shown for $a = x_0 - \delta_0$, $b = x_0 + \delta_0$ and $c = \frac{1}{2} f(x_0)$ that $x \in (a, b)$ implies $f(x) > c$. □

[5.] *Let $f : \mathbb{R} \to \mathbb{R}$ and $a, L \in \mathbb{R}$. State the definition: $\lim_{x \to a} f(x) = L$. Using just the definition and not the limit theorems, show that $\lim_{x \to 1} (x + 3)^2 = 16$.*

Definition. $\lim_{x \to a} f(x) = L$ means for every $\epsilon > 0$ there is $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x \in \mathbb{R}$ and $0 < |x - a| < \delta$.

Proof. Choose $\epsilon > 0$. Let $\delta = \min\{1, \frac{\epsilon}{9}\}$. For any $x \in \mathbb{R}$ that satisfies $0 < |x - 1| < \delta$ we have $|x + 7| = |x - 1 + 8| \leq |x - 1| + 8 < \delta + 8 \leq 1 + 8 = 9$ because $\delta \leq 1$. Hence, $\delta \leq \frac{\epsilon}{9}$ implies

$$|f(x) - 16| = |(x + 3)^2 - (1 + 3)^2| = |(x + 3 + 1 + 3)(x + 3 - 1 - 3)| = |x + 7||x - 1| < 9\delta \leq \epsilon. \quad \square$$