Math 3210 § 2. **Third Midterm Exam** Name: SomeSolutions
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More Problems.

1. *Show that the sinc function* $f:[0,1]\to\mathbb{R}$ *is continuous, where* $f(x)=\begin{cases}\dfrac{\sin x}{x}, & \textit{if } x>0,\\ 1, & \textit{if } x=0.\end{cases}$

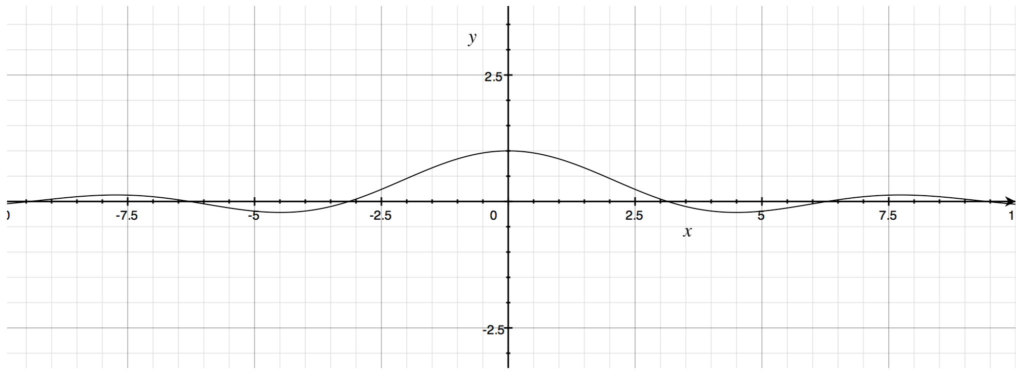

Figure 1: Sinc Function

Proof. We shall take as our starting point geometric inequalities satisfied by sine. $P=(\cos x,\sin x)$ is the coordinate of a point on the unit circle. The horizontal distance of P to the y-axis is at most one so $|\cos x|\le 1$. The distance of P to the x-axis is the vertical distance, which is $|\sin x|$. This is less than the distance around the circle from $(1,0)$ which is $|x|$. On the other hand, for $|x|<\frac{\pi}{2}$, the shortest curve from the positive x-axis to the ray $\overrightarrow{OP}$ outside the unit circle is the arc of the circle of length $|x|$, which is less than the vertical path above $(1,0)$ which has length $|\tan x|$. Thus, for $|x|<\frac{\pi}{2}$

$$|\sin x|\le|x|\le|\tan x|=\frac{|\sin x|}{|\cos x|}=\frac{|\sin x|}{\sqrt{1-\sin^2 x}}.$$

Multiplying the last inequality

$$(1-\sin^2 x)x^2\le\sin^2 x$$

so

$$x^2\le(1+x^2)\sin^2 x.$$

Thus for $0 < x < \frac{\pi}{2}$,

$$\begin{aligned} 1 \geq \frac{\sin x}{x} &\geq \frac{1}{\sqrt{1+x^2}} \\ &= 1 + \frac{1-\sqrt{1+x^2}}{\sqrt{1+x^2}} \\ &= 1 + \frac{(1-\sqrt{1+x^2})(1+\sqrt{1+x^2})}{\sqrt{1+x^2}(1+\sqrt{1+x^2})} \\ &= 1 + \frac{1-(1+x^2)}{\sqrt{1+x^2}(1+\sqrt{1+x^2})} \\ &= 1 - \frac{x^2}{\sqrt{1+x^2}(1+\sqrt{1+x^2})} \\ &\geq 1 - \frac{x^2}{\sqrt{1+0^2}(1+\sqrt{1+0^2})} \\ &= 1 - \frac{x^2}{2} \end{aligned} \tag{1}$$

Thus for $0 < x < \frac{\pi}{2}$,

$$\left| \frac{\sin x}{x} - 1 \right| \leq \frac{x^2}{2}. \tag{2}$$

The inequality between sines at two numbers will follow later on in the course from knowing that sine has bounded derivative, or that it is the integral of a bounded function. For now we will content ourselves with the inequality above and trig identities. For $x, y \in \mathbb{R}$, using the addition formulae,

$$\begin{aligned} \sin x - \sin y &= \sin\left(\tfrac{x+y}{2} + \tfrac{x-y}{2}\right) - \sin\left(\tfrac{x+y}{2} - \tfrac{x-y}{2}\right) \\ &= \sin\left(\tfrac{x+y}{2}\right)\cos\left(\tfrac{x-y}{2}\right) + \cos\left(\tfrac{x+y}{2}\right)\sin\left(\tfrac{x-y}{2}\right) \\ &\quad - \sin\left(\tfrac{x+y}{2}\right)\cos\left(\tfrac{x-y}{2}\right) + \cos\left(\tfrac{x+y}{2}\right)\sin\left(\tfrac{x-y}{2}\right) \\ &= 2\cos\left(\tfrac{x+y}{2}\right)\sin\left(\tfrac{x-y}{2}\right) \end{aligned}$$

Thus for $x, y \in \mathbb{R}$,

$$|\sin x - \sin y| \leq 2\left|\cos\left(\tfrac{x+y}{2}\right)\right| \left|\sin\left(\tfrac{x-y}{2}\right)\right| \leq 2 \cdot 1 \cdot \left|\tfrac{x-y}{2}\right| = |x-y|. \tag{3}$$

Finally, the inequality between sinc functions at different points follows by sneaking in a cross term. For $0 < x, y$, by (3),

$$\begin{aligned} \left|\tfrac{\sin x}{x} - \tfrac{\sin y}{y}\right| &= \left|\tfrac{\sin x}{x} - \tfrac{\sin x}{y} + \tfrac{\sin x}{y} - \tfrac{\sin y}{y}\right| \\ &\leq \left|\tfrac{\sin x}{x} - \tfrac{\sin x}{y}\right| + \left|\tfrac{\sin x}{y} - \tfrac{\sin y}{y}\right| \\ &= |\sin x| \left|\tfrac{1}{x} - \tfrac{1}{y}\right| + \tfrac{1}{|y|}|\sin x - \sin y| \\ &= |\sin x| \left|\tfrac{y-x}{xy}\right| + \tfrac{1}{|y|}|\sin x - \sin y| \\ &\leq \tfrac{|\sin x|}{|x|}\tfrac{|y-x|}{|y|} + \tfrac{1}{|y|}|x-y| \end{aligned}$$

Hence for $0 < x, y$,

$$\left|\tfrac{\sin x}{x} - \tfrac{\sin y}{y}\right| \leq \tfrac{2}{|y|}|y-x|. \tag{4}$$

Now the main part of the proof may begin. Choose $a \in [0,1]$. We argue two cases: $a = 0$ and $a > 0$ seperately.

In case $a = 0$, choose $\varepsilon > 0$. Let $\delta = \sqrt{2\varepsilon}$. If $x \in [0,1]$ such that $|a - x| < \delta$, then if $0 < x$, by (2),

$$|f(x) - f(a)| = \left|\tfrac{\sin x}{x} - 1\right| \le \tfrac{x^2}{2} < \tfrac{(\sqrt{2\varepsilon})^2}{2} = \varepsilon.$$

If $x = 0$ then $|f(x) - f(a)| = |1 - 1| = 0 < \varepsilon$. Thus for every $x \in [0,1]$ with $|a - x| < \delta$ we have $|f(x) - f(a)| < \varepsilon$, completing the proof that f is continuous at $a = 0$.

In case $a > 0$, choose $\varepsilon > 0$. Let $\delta = \min\{a, \frac{a\varepsilon}{2}\}$. Then choose $x \in [0,1]$ so that $|x - a| < \delta$. But this implies $x = a + x - a \ge x - |a - x| > a - a = 0$ so $x > 0$ also. By (4),

$$|f(x) - f(a)| = \left|\tfrac{\sin x}{x} - \tfrac{\sin a}{a}\right| \le \tfrac{2}{|a|}\,|x - a| < \tfrac{2}{|a|}\,\tfrac{a\varepsilon}{2} = \varepsilon,$$

completing the proof that f is continuous at $a > 0$. □

2. *Show that the sinc function* $g(x) = \dfrac{\sin x}{x}$ *is uniformly continuous on* $(0,1)$.

We observe that the function $f : [0,1] \to \mathbb{R}$ from Problem (1) is an extension of g, *i.e.*, $f(x) = g(x)$ for all $x \in (0,1)$. We showed there that f is continuous on $[0,1]$. By the theorem that say that any function $f : I \to \mathbb{R}$ that is continuous on a closed and bounded interval is also uniformly continuous, we have that f is uniformly continuous on $I = [0,1]$. But if a function is uniformly continuous on a set, it is automatically uniformly continuous on a subset. Thus f is uniformly continuous on $(0,1)$. But $g = f$ when restricted to $(0,1)$, so g is uniformly continuous on $(0,1)$.

3. *Using only the definition of uniform continuity, show that the sinc function* $g(x) = \dfrac{\sin x}{x}$ *is uniformly continuous on* $(0,1)$.

The continuity proof from problem (1) cannot be used because the δ there depends on a and tends to zero as $a \to 0$. If δ had a positive minimum on $[0,1]$ then that would prove the uniform continuity. In fact, by using our inequalities more carefully, we can recover a uniform δ rather like the proof that $\sqrt{x}$ is uniformly continuous on $[0,\infty)$.

To begin the proof, choose $\varepsilon > 0$. Let $\delta = \min\{\frac{\sqrt{\varepsilon}}{2}, \frac{\varepsilon^{3/2}}{4}\}$. Now choose $x, y \in (0,1)$ such that $|x - y| < \delta$. One of the two numbers is smaller, so after swapping if necessary, we may suppose that $x \le y$. The argument will done in two parts: in case $x < \frac{\sqrt{\varepsilon}}{2}$ or in case $x \ge \frac{\sqrt{\varepsilon}}{2}$.

In case $x < \frac{\sqrt{\varepsilon}}{2}$, we have $y = x + y - x \le x + |y - x| < x + \delta \le \frac{\sqrt{\varepsilon}}{2} + \frac{\sqrt{\varepsilon}}{2} = \sqrt{\varepsilon}$. The inequalities (1) say

$$1 - \frac{\varepsilon}{8} < 1 - \frac{x^2}{2} \le \frac{\sin x}{x} < 1,$$
$$-1 < -\frac{\sin y}{y} \le -1 + \frac{y^2}{2} < -1 + \frac{\varepsilon}{2}.$$

Adding

$$-\frac{\varepsilon}{8} < \frac{\sin x}{x} - \frac{\sin y}{y} < \frac{\varepsilon}{2}$$

which implies

$$\left|\frac{\sin x}{x} - \frac{\sin y}{y}\right| < \varepsilon.$$

In case $x \geq \frac{\sqrt{\varepsilon}}{2}$ so also $y \geq x \geq \frac{\sqrt{\varepsilon}}{2}$ we use (4) instead.

$$\left|\frac{\sin x}{x} - \frac{\sin y}{y}\right| \leq \frac{2}{|y|}\,|y-x| < \frac{4\delta}{\sqrt{\varepsilon}} \leq \frac{4}{\sqrt{\varepsilon}} \cdot \frac{\varepsilon^{3/2}}{4} = \varepsilon.$$

Thus we have shown in both cases that if $x, y \in (0,1)$ such that $|x-y| < \delta$ then

$$|g(x) - g(y)| = \left|\frac{\sin x}{x} - \frac{\sin y}{y}\right| < \varepsilon.$$

hence $g(x)$ is uniformly continuous on $(0,1)$. □