Math 3210 § 1. **Practice Problems for Final** Name: Solutions
Treibergs April 22, 2015

The final will be half over the material since the last midterm (differentiation, integration and series) and half comprehensive. These additional problems deal with infinite series, that are not well represented in previous problem sets and exams.

---

1. *State the definition: The series $\sum_{k=0}^{\infty} a_k$ converges. Using only the definition, determine whether the following series converge or diverge.*

   (a) $1 - 3 + 5 - 7 + \cdots + (-1)^{n+1}(2n-1) + \cdots$

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{\lfloor \frac{(n+1)}{2} \rfloor}$

   (c) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right)$

   An infinite series $\sum_{k=0}^{\infty} a_k$ is said to converge to $S \in \mathbf{R}$ iff the limit of partial sums exists and converges to S:
   $$S = \lim_{n\to\infty} S_n \qquad \text{where} \qquad S_n = \sum_{k=0}^{n} a_k.$$

   (a) Computing the partial sum for the first series, we get by induction that
   $$S_1 = 1, \qquad S_2 = 1 - 3 = -1, \qquad S_3 = 1 - 3 + 5 = 1, \ldots, \qquad S_n = (-1)^{n+1}, \ldots$$
   The series DOES NOT CONVERGE since the limit of partial sums
   $$\lim_{n\to\infty} (-1)^{n+1}$$
   does not converge.

   (b) The partial sums for the second series are
   $$S_1 = \frac{1}{1} = 1, \quad S_2 = \frac{1}{1} - \frac{1}{1} = 0, \quad S_3 = \frac{1}{1} - \frac{1}{1} + \frac{1}{2} = \frac{1}{2}, \quad S_4 = \frac{1}{1} - \frac{1}{1} + \frac{1}{2} - \frac{1}{2} = 0, \ldots$$
   By induction we can show that
   $$S_n = \begin{cases} \dfrac{2}{n+1}, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$
   The infinite series (b) CONVERGES to S where
   $$S = \lim_{n\to\infty} S_n = 0.$$

   (c) The partial sums of the third series telescope
   $$S_n = \sum_{k=1}^{n} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right) = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{n+1}}$$
   so that the series CONVERGES to S where
   $$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{n+1}} \right) = 1.$$

2. *Express the repeating decimal fraction $\alpha = .\overline{423}$ as an infinite series. Determine whether the series converges. If so, find its sum.*

The decimal fraction repeats $\alpha = .423423423\ldots$ which means it is the sum of 423 times appropriate powers of 1000.

$$\alpha = \sum_{k=1}^{\infty} 423 \left(\frac{1}{1000}\right)^k = \sum_{k=1}^{\infty} ar^k$$

which is a geometric series with $a = 423$ and $r = 10^{-3}$. The partial sum equals the decimal cut off at $3n$ digits,

$$\overbrace{.423\ldots423}^{3n} = S_n = ar\sum_{k=0}^{n-1} r^k = ar\frac{1-r^n}{1-r} = \frac{423}{1000} \cdot \frac{1 - \frac{1}{1000^n}}{1 - \frac{1}{1000}}$$

which tends to

$$\frac{423}{1000} \cdot \frac{1}{1 - \frac{1}{1000}} = \frac{423}{999} = \frac{141}{333}$$

as $n \to \infty$, thus the sum converges.

3. *Prove the Limit Comparison Test: Suppose $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are two series of positive terms such that the sequence $\{a_k/b_k\}$ converges to a positive limit. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.*

Suppose

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lambda > 0.$$

From the definition of limit, there is $n_0 \in \mathbf{N}$ so that

$$\left|\frac{a_k}{b_k} - \lambda\right| < \frac{\lambda}{2} \qquad \text{whenever } k > n_0.$$

Thus

$$\frac{\lambda}{2} \le -\frac{\lambda}{2} + \lambda \le -\left|\frac{a_k}{b_k} - \lambda\right| + \lambda \le \frac{a_k}{b_k} = \left(\frac{a_k}{b_k} - \lambda\right) + \lambda \le \left|\frac{a_k}{b_k} - \lambda\right| + \lambda < \frac{\lambda}{2} + \lambda = \frac{3\lambda}{2}.$$

Now we apply the comparison test. If $A = \sum_{k=1}^{\infty} a_k$ converges, then for $k > n_0$

$$b_k \le \frac{2}{\lambda} a_k$$

so that $\sum_{k=1}^{\infty} b_k$ converges. To see this more closely, if suffices to show that the partial sums $B_n = \sum_{k=1}^{n} b_k$ which increase with n, are bounded. For $n > n_0$, by the inequality,

$$B_n = \sum_{k=1}^{n} b_k = \sum_{k=1}^{n_0} b_k + \sum_{k=n_0+1}^{n} b_k \le \sum_{k=1}^{n_0} b_k + \frac{2}{\lambda}\sum_{k=n_0+1}^{n} a_k \le \sum_{k=1}^{n_0} b_k + \frac{2}{\lambda}\sum_{k=1}^{n} a_k \le \sum_{k=1}^{n_0} b_k + \frac{2A}{\lambda}.$$

To see the converse, we argue similarly. Assuming $B = \sum_{k=1}^{\infty} b_k$ converges, then for $k > n_0$

$$a_k \le \frac{3\lambda}{2} b_k$$

This implies for $n > n_0$ the partial sums are bounded

$$A_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n_0} a_k + \sum_{k=n_0+1}^{n} a_k \le \sum_{k=1}^{n_0} a_k + \frac{3\lambda}{2}\sum_{k=n_0+1}^{n} b_k \le \sum_{k=1}^{n_0} a_k + \frac{3\lambda B}{2}.$$

so that the partial are bounded, hence $\lim_{n\to\infty} A_n$ exists.

4. *For what value p does the series* $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\log k)^p}$ *converge?*

If $p \le 0$ then for $k \ge 3$, since $\log k \ge 1$ we have

$$\frac{1}{k(\log k)^p} \ge \frac{1}{k}$$

so that by comparing sums,

$$\sum_{k=3}^{n} \frac{1}{k(\log k)^p} \ge \sum_{k=3}^{n} \frac{1}{k}$$

which is the harmonic series that diverges as $n \to \infty$.

Because

$$f(x) = \frac{1}{x(\log x)^p}, \qquad f'(x) = f(x) = -\frac{(p + \log x)(\log x)^{p-1}}{x^2(\log x)^{2p}}$$

so for $p > 0$ we have $f'(x) < 0$ so $f(x)$ is decreasing for $x \ge 2$ so we may apply the Integral Test. Substitute $u = \log x$ and $du = dx/x$ to find

$$I = \int_2^R \frac{dx}{x(\log x)^p} = \int_{\log 2}^{\log R} \frac{du}{u^p}$$

For $p = 1$,

$$I = \log(\log R) - \log(\log 2)$$

which diverges as $R \to \infty$. For $p \ne 1$,

$$I = \left[\frac{u^{1-p}}{1-p}\right]_{\log 2}^{\log R} = \frac{1}{1-p}\left(\frac{(\log R)^{1-p}}{1-p} - \frac{(\log 2)^{1-p}}{1-p}\right)$$

which converges as $R \to \infty$ if and only if $p > 1$. The Integral Test says the series converges exactly when the improper integral converges, which occurs if and only if $p > 1$.

5. *Determine whether each of the following series converges absolutely, converges conditionally or diverges.*

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^n n}{n^2 - 5n + 1}$

   (b) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{\sqrt{k+1} - \sqrt{k}}{k}$

   (c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k (k!)^2}{(2k)!}$

   (a) Converges Conditionally. The larger zero of $x^2 - 5x + 1$ is $5/2 + \sqrt{21}/2 = 4.791$ so that

$$f(x) = \frac{x}{x^2 - 5x + 1}$$

is decreasing to zero on $[5, \infty)$ so that the alternating series test may be applied to $\displaystyle\sum_{k=5}^{\infty} \frac{(-1)^n n}{n^2 - 5n + 1}$.

It follows that (a) converges. However, for $x \ge 5 > 4.791$, the denominator is positive and smaller than $x^2 + 1$ so

$$f(x) = \frac{x}{x^2 - 5x + 1} \ge \frac{x}{x^2 + 1}$$

and
$$\int_5^R \frac{x\,dx}{x^2-5x+1} \geq \int_5^R \frac{x\,dx}{x^2+1} = \frac{\log(R^2+1) - \log 26}{2}$$
tends to infinity as $R \to \infty$. Hence the absolute series $\displaystyle\sum_{k=5}^{\infty} \frac{n}{n^2-5n+1}$ fails to converge, implying that (a) is not absolutely convergent.

(b) CONVERGES ABSOLUTELY. We see that for $n \in \mathbf{N}$,
$$\frac{\sqrt{k+1}-\sqrt{k}}{k} = \frac{\left(\sqrt{k+1}-\sqrt{k}\right)\left(\sqrt{k+1}+\sqrt{k}\right)}{k\left(\sqrt{k+1}+\sqrt{k}\right)} = \frac{1}{k\left(\sqrt{k+1}+\sqrt{k}\right)} \leq \frac{1}{k^{3/2}}.$$
Hence by the comparison test the absolute series
$$\sum_{k=1}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{k}$$
converges because the p-series
$$\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$$
converges since $p > 1$.

(c) CONVERGES ABSOLUTELY. We use the ratio test on the absolute series.
$$\frac{|a_{k+1}|}{|a_k|} = \frac{\dfrac{([k+1]!)^2}{(2[k+1])!}}{\dfrac{(k!)^2}{(2k)!}} = \frac{([k+1]!)^2}{(k!)^2} \cdot \frac{(2k)!}{(2[k+1])!} = \frac{(k+1)^2}{(2k+2)(2k+1)} \to \frac{1}{4} = \rho$$
as $k \to \infty$. Since $\rho < 1$ the absolute series $\displaystyle\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$ is convergent.

6. *Assume that* $\left|\dfrac{a_{k+1}}{a_k}\right| \leq \dfrac{k^2}{(k+1)^2}$ *for all* $k \in \mathbf{N}$. *prove that the series* $\displaystyle\sum_{k=1}^{\infty} a_k$ *converges absolutely.*

Observe that from the assumption
$$\begin{aligned} |a_{k+1}| &= \left|\frac{a_{k+1}}{a_k}\right| \cdot \left|\frac{a_k}{a_{k-1}}\right| \cdot \left|\frac{a_{k-1}}{a_{k-2}}\right| \cdots \left|\frac{a_2}{a_1}\right| \cdot |a_1| \\ &\leq \frac{k^2}{(k+1)^2} \cdot \frac{(k-1)^2}{k^2} \cdot \frac{(k-2)^2}{(k-1)^2} \cdots \frac{1^2}{2^2} \cdot |a_1| = \frac{|a_1|}{(k+1)^2} \end{aligned}$$
Now by the comparison theorem, the absolute series is dominated by
$$\sum_{k=1}^{n} |a_k| \leq |a_1| + \sum_{k=2}^{n} \frac{|a_1|}{k^2}$$
which converges as $n \to \infty$ because, by the integral test, the improper integral converges
$$\int_2^{\infty} \frac{dx}{x^2} = \frac{1}{2}.$$