Math 3210 § 1. **Second Midterm Exam** Name: Solutions
Treibergs March 4, 2015

1. *Let $\{a_n\}$ be a real sequence and $L \in \mathbf{R}$. State the definition: $a_n \to L$ as $n \to \infty$. Find the limit. Using just the definition, prove that your answer is correct.*

$$L = \lim_{n\to\infty} \frac{(2n+1)^2}{n^2+n-19}.$$

The real sequence $\{a_n\}$ is said to converge to $L \in \mathbf{R}$ if for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ so that

$$|a_n - L| < \varepsilon \qquad \text{whenever} \qquad n > N.$$

Using the main theorem about limits, we can determine the limit.

$$L = \lim_{n\to\infty} \frac{(2n+1)^2}{n^2+n-19} = \lim_{n\to\infty} \frac{\left(2+\frac{1}{n}\right)^2}{1+\frac{1}{n}-\frac{19}{n^2}} = \frac{\lim\limits_{n\to\infty}\left(2+\frac{1}{n}\right)^2}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}-\frac{19}{n^2}\right)}$$

$$= \frac{\left(\lim\limits_{n\to\infty}\left[2+\frac{1}{n}\right]\right)^2}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty}\frac{1}{n} - \lim\limits_{n\to\infty}\frac{19}{n^2}} = \frac{\left(\lim\limits_{n\to\infty} 2 + \lim\limits_{n\to\infty}\frac{1}{n}\right)^2}{1+0-0} = (2+0)^2 = 4.$$

To give a proof using only the definition, choose $\varepsilon > 0$. Let $N = \max\left\{19, \sqrt{\frac{77}{\varepsilon}}\right\}$. Then for any $n \in \mathbf{N}$ such that $n > N$ we have since $n > N \geq 19$ that

$$|a_n - L| = \left|\frac{(2n+1)^2}{n^2+n-19} - 4\right| = \frac{|(4n^2+4n+1)-(4n^2+4n-76)|}{n^2+n-19} \leq \frac{77}{n^2} < \frac{77}{N^2} \leq \varepsilon.$$

2. *Suppose that $\{a_n\}$ is a real sequence and $\alpha, \beta \in \mathbf{R}$. Suppose that the sequence converges $a_n \to \alpha$ as $n \to \infty$ and the terms satisfy $a_n \leq \beta$ for all $n \in \mathbf{N}$.* Show that $\alpha \leq \beta$.

We will show that for every $\varepsilon > 0$ we have $\alpha < \beta + \varepsilon$ from which $\alpha \leq \beta$ follows. Let $\varepsilon > 0$ be arbitrary. By the convergence $a_n \to \alpha$ as $n \to \infty$, there is an $N \in \mathbf{R}$ such that

$$|a_n - \alpha| < \varepsilon \qquad \text{whenever } n > N. \tag{1}$$

By the Archimedean Property, there is an $n_0 \in \mathbf{N}$ such that $n_0 > N$. For this n_0 we have using $a_{n_0} \leq \beta$ and (1),

$$\alpha = a_{n_0} + (\alpha - a_{n_0}) \leq a_{n_0} + |\alpha - a_{n_0}| < \beta + \varepsilon$$

as to be shown.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT: *Let $f, g : \mathbf{R} \to \mathbf{R}$ be two bounded functions. Then* $\sup_{\mathbf{R}}(f-g) = \sup_{\mathbf{R}} f - \inf_{\mathbf{R}} g$.

   FALSE. Take $f(x) = g(x) = \sin x$ which are bounded functions. Then $f - g = 0$ so the left side is zero. On the other hand $\sup_{\mathbf{R}} f = 1$ and $\inf_{\mathbf{R}} g = -1$ so that the right side is $1 - (-1) = 2 \ne 0$.

   (b) STATEMENT: *Suppose that $\{a_n\}$ is a real sequence which has no convergent subsequence. Then $\{a_n\}$ is unbounded.*

   TRUE. This is the contrapositive of the Bolzano Weierstrass Theorem: Let $\{a_n\}$ be a real sequence. If $\{a_n\}$ is bounded, then $\{a_n\}$ has a convergent subsequence.

   (c) STATEMENT: *Suppose for each $n \in \mathbf{N}$ there are real numbers a_n and b_n such that $0 \le a_n < b_n \le 1$ and that $b_n - a_n = \dfrac{1}{2^n}$. Then $\bigcap_{n\in\mathbf{N}} [a_n, b_n] \ne \emptyset$.*

   FALSE. Take $I_1 = \left[0, \dfrac{1}{2}\right]$, $I_2 = \left[\dfrac{3}{4}, 1\right]$ and $I_n = \left[0, \dfrac{1}{2^n}\right]$ for $n \ge 3$. Then $I_1 \cap I_2 = \emptyset$ so $\bigcap_{n\in\mathbf{N}} I_n = \emptyset$. The Nested Intervals Theorem does not apply since the hypothesis of nested is missing.

4. *Let b_n be a sequence of -1's, 0's and 1's and define the numbers*

$$a_n = \sum_{k=1}^{n} \frac{b_k}{3^k}.$$

State the definition: $\{a_n\}$ is a Cauchy Sequence. *Prove that the sequence $\{a_n\}$ converges.*

The real sequence $\{a_n\}$ is a *Cauchy Sequence* if for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|a_n - a_\ell| < \varepsilon \qquad \text{whenever } n > N \text{ and } \ell > N.$$

To show that the given $a_n \to a$ as $n \to \infty$ for some $a \in \mathbf{R}$ we show that $\{a_n\}$ is a Cauchy Sequence, thus is convergent. Choose $\varepsilon > 0$. Let $N = -\dfrac{\log(2\varepsilon)}{\log 3}$. For any $n, \ell \in \mathbf{R}$ such that $n > N$ and $\ell > N$ we have either $\ell = n$ in which case $|a_n - a_\ell| = 0 < \varepsilon$ or $\ell \ne n$. By swapping the roles of ℓ and n if necessary, we may assume that $\ell < n$. Then since $|b_k| \le 1$ for all k and $\ell > N$,

$$\begin{aligned}
|a_n - a_\ell| &= \left| \sum_{k=1}^{n} \frac{b_k}{3^k} - \sum_{k=1}^{\ell} \frac{b_k}{3^k} \right| = \left| \sum_{k=\ell+1}^{n} \frac{b_k}{3^k} \right| \le \sum_{k=\ell+1}^{n} \frac{|b_k|}{3^k} \le \sum_{k=\ell+1}^{n} \frac{1}{3^k} \\
&= \frac{\left(\frac{1}{3}\right)^{\ell+1} - \left(\frac{1}{3}\right)^{n+1}}{1 - \frac{1}{3}} \le \frac{3}{2}\left(\frac{1}{3}\right)^{\ell+1} = \frac{1}{2\cdot 3^\ell} < \frac{1}{2\cdot 3^N} = \varepsilon
\end{aligned}$$

where we have used $\displaystyle\sum_{k=\ell+1}^{n} r^k = \frac{r^{\ell+1} - r^{n+1}}{1-r}$.

5. *Let the real sequence be defined recursively. Show that the sequence $\{a_n\}$ is bounded above. Show that there is $L \in \mathbf{R}$ so that the sequence converges $a_n \to L$ as $n \to \infty$. Find L.*

$$a_1 = 1, \qquad a_{n+1} = \frac{2a_n + 4}{5} \text{ for all } n \in \mathbf{N}.$$

We show that the sequence is bounded above, $a_n \le M = 2$ for all n, by induction. (Any $M \ge \frac{4}{3}$ will work.) In the base case $a_1 = 1 \le 2$ holds. In the induction case, we assume that for some $n \in \mathbf{N}$ we have $a_n \le 2$. Then by the recursion

$$a_{n+1} = \frac{2}{5}a_n + \frac{4}{5} \le \frac{2}{5} \cdot 2 + \frac{4}{5} = \frac{8}{5} \le 2.$$

Thus the induction case holds as well. By mathematical induction, $a_n \le 2$ for all $n \in \mathbf{N}$.

To show that $\{a_n\}$ converges to some $L \in \mathbf{R}$, we show that a_n is increasing. Since it is also bounded above, $L = \lim_{n\to\infty} a_n$ exists by the Monotone Convergence Theorem. To see that a_n is increasing we show that $a_{n+1} > a_n$ for all $n \in \mathbf{N}$ using induction. In the base case,

$$a_2 = \frac{2}{5}a_1 + \frac{4}{5} = \frac{2}{5} \cdot 1 + \frac{4}{5} = \frac{6}{5}$$

so $a_2 > a_1 = 1$. In the induction case, we assume that $a_{n+1} - a_n > 0$ for some $n \in \mathbf{N}$. By the recursion and induction hypothesis

$$a_{n+2} - a_{n+1} = \frac{2}{5}a_{n+1} + \frac{4}{5} - \left(\frac{2}{5}a_n + \frac{4}{5}\right) = \frac{2}{5}(a_{n+1} - a_n) > 0.$$

Hence the induction step holds as well, and we conclude that $a_{n+1} > a_n$ for all $n \in \mathbf{N}$ by mathematical induction.

To find the limiting value, we pass the recursion equation to the limit.

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(\frac{2}{5}a_n + \frac{4}{5}\right) = \frac{2}{5}L + \frac{4}{5}.$$

Solving yields $L = \dfrac{4}{3}$.