1. *Let $f:\mathbf{R}\to\mathbf{R}$. State the definition: f is continuous on $\mathbf{R}$. Using just the definition, prove that $f(x)=x^4$ is continuous on $\mathbf{R}$. Is $f(x)=x^4$ uniformly continuous on $\mathbf{R}$? Give a SHORT explanation.*

   A function $f:\mathbf{R}\to\mathbf{R}$ is said to be *continuous* on $\mathbf{R}$ is it is continuous at every $a\in\mathbf{R}$.

   f is continuous at a means for every $\epsilon>0$ there is a $\delta>0$ such that
$$|f(x)-f(a)|<\epsilon \qquad \text{whenever } x\in\mathbf{R} \text{ and } |x-a|<\delta.$$
   To show $f(x)=x^4$ is continuous on $\mathbf{R}$, choose $a\in\mathbf{R}$ and $\epsilon>0$. Let
$$\delta=\min\left\{1,\frac{\epsilon}{\left((|a|+1)^2+|a|^2\right)(1+2|a|)}\right\}.$$
   Then for any $x\in\mathbf{R}$ such that $|x-a|<\delta$, because $\delta\le 1$ we have
$$|x|=|a+x-a|\le|a|+|x-a|\le|a|+1.$$
   Because also $\delta\le\dfrac{\epsilon}{\left((|a|+1)^2+|a|^2\right)(1+2|a|)}$ we have
$$\begin{aligned}|f(x)-f(a)|&=\left|x^4-a^4\right|=\left|(x^2+a^2)(x^2-a^2)\right|=\left|(x^2+a^2)(x+a)(x-a))\right|\\&=\left|x^2+a^2\right|\,|x+a|\,|x-a|\le\left(|x|^2+|a|^2\right)(|x|+|a|)\,|x-a|\\&\le\left((|a|+1)^2+|a|^2\right)(|a|+1+|a|)\,|x-a|\\&<\left((|a|+1)^2+|a|^2\right)(1+2|a|)\,\delta\\&\le\left((|a|+1)^2+|a|^2\right)(1+2|a|)\,\frac{\epsilon}{\left((|a|+1)^2+|a|^2\right)(1+2|a|)}=\epsilon.\end{aligned}$$
   f is NOT UNIFORMLY CONTINUOUS because δ depends on both ϵ and a. If f were uniformly continuous, δ would depend only on ϵ.

2. *Let $f,f_n:\mathbf{R}\to\mathbf{R}$ be functions. State the definition: the sequence of functions $\{f_n\}$ converges uniformly on $\mathbf{R}$ to a function f. Let $f_n(x)=\dfrac{x}{n^2+x^2}$. Determine whether there is a function $f(x)$ such that $\{f_n\}$ converges uniformly to f, converges pointwise but not uniformly to f or does not converge to any f on $\mathbf{R}$. Prove your result.*

   We say that a sequence of functions $\{f_n\}$ *converges uniformly* to a function f on $\mathbf{R}$ if for every $\epsilon>0$ there is an $N\in\mathbf{R}$ such that
$$|f_n(x)-f(x)|<\epsilon \qquad \text{whenever } x\in\mathbf{R} \text{ and } n>N.$$
   The sequence $f_n(x)=\dfrac{x}{n^2+x^2}$ CONVERGES UNIFORMLY to the function $f(x)=0$ on $\mathbf{R}$. For each $x\in\mathbf{R}$ we have the limit
$$\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\frac{x}{n^2+x^2}=0$$
   thus $\{f_n\}$ converges pointwise to the function $f(x)=0$. To see that the convergence is uniform, choose $\epsilon>0$ and let $N=1/\epsilon$. Then if $x\in\mathbf{R}$ and $n>N$ we have
$$\begin{aligned}|f_n(x)-f(x)|&=\left|\frac{x}{n^2+x^2}-0\right|=\frac{|x|}{n^2+x^2}=\frac{\sqrt{x^2}}{n^2+x^2}\\&\le\frac{\sqrt{n^2+x^2}}{n^2+x^2}=\frac{1}{\sqrt{n^2+x^2}}\le\frac{1}{\sqrt{n^2}}=\frac{1}{n}<\frac{1}{N}=\epsilon.\end{aligned}$$

Alternately, we can use the method suggested in your homework to deduce the inequality. Observing that $f_n(x) \to 0$ as $x \to \pm\infty$, all we need to do is find the maximum and minimum values of f_n. Differentiating,

$$\frac{d}{dx} f_n(x) = \frac{d}{dx}\left(\frac{x}{n^2+x^2}\right) = \frac{n^2-x^2}{(n^2+x^2)^2}$$

so that $x = \pm n$ are the only critical points corresponding to maximum and minimum. Thus

$$-\frac{1}{2n} = -\frac{n}{n^2+n^2} = f_n(-n) \le f_n(x) = \frac{x}{n^2+x^2} \le f_n(n) = \frac{n}{n^2+n^2} = \frac{1}{2n}$$

so $|f_n(x) - f(x)| < 1/n$ as above.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) *Suppose that $f : (-1, 1) \to \mathbf{R}$ is a differentiable strictly increasing function such that $f(0) = 2$. Then the inverse function f^{-1} is differentiable at $y = 2$*

   FALSE. The inverse function is differentiable if also $f'(0) > 0$. For example $f(x) = 2 + x^3$ is strictly increasing, $f(0) = 2$, but the inverse function $f^{-1}(y) = \sqrt[3]{y-2}$ is not differentiable at $y = 2$.

   (b) *The function $f(x) = \dfrac{x^3 - 2x^2 - 3x - 4}{5x^2 + 6}$ has a real root.*

   TRUE. The denominator is always positive, so the rational function $f(x)$ is continuous on $\mathbf{R}$. If $f(x) > 0$ and $f(y) < 0$ then zero is intermediate so by the Intermediate Value Theorem, $f(c) = 0$ for some c is between x and y. Note that $f(0) = -\dfrac{2}{3}$ and $f(10) = \dfrac{1000 - 2 \cdot 100 - 3 \cdot 10 - 4}{5 \cdot 100 + 6} = \dfrac{766}{506} > 0$. Thus some $c \in (0, 10)$ is a root of f.

   (c) *Suppose that $f, g : (a, b) \to \mathbf{R}$ are differentiable functions such that $g(x) \neq 0$ for all x and such that the finite limit $\lim\limits_{x \to a+} \dfrac{f'(x)}{g'(x)} = L$ exists. Then $\lim\limits_{x \to a+} \dfrac{f(x)}{g(x)} = M$ exists and $L = M$.*

   FALSE. L'Hopital's Theorem does not apply since this is neither the "$\frac{0}{0}$" nor the "$\frac{\infty}{\infty}$" case. Take $f(x) = x - b$ and $g(x) = x - a$ so $g(x) > 0$ on $x \in (a, b)$, then $\lim\limits_{x \to a+} \dfrac{f'(x)}{g'(x)} = \lim\limits_{x \to a+} \dfrac{1}{1} = 1$ exists but $\lim\limits_{x \to a+} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a+} \dfrac{x-b}{x-a} = -\infty$ does not have the same limit.

4. *Let $f : \mathbf{R} \to \mathbf{R}$ be a function and $a \in \mathbf{R}$ a point. State the definition: f is differentiable at a. Using just the definition of differentiable and not differentiation rules, show that $f(x) = \dfrac{x}{1 + x + x^2}$ is differentiable at $a \in \mathbf{R}$.*

   $f : \mathbf{R} \to \mathbf{R}$ is said to be *differentiable* at $a \in \mathbf{R}$ if there is a real number $f'(a)$ such that

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

For this f, for $x \neq a$

$$\frac{f(x)-f(a)}{x-a} = \frac{\dfrac{x}{1+x+x^2} - \dfrac{a}{1+a+a^2}}{x-a} = \frac{x(1+a+a^2)-a(1+x+x^2)}{(1+x+x^2)(1+a+a^2)(x-a)}$$
$$= \frac{x-a+a^2x-ax^2}{(1+x+x^2)(1+a+a^2)(x-a)} = \frac{(1-ax)(x-a)}{(1+x+x^2)(1+a+a^2)(x-a)}$$
$$= \frac{(1-ax)}{(1+x+x^2)(1+a+a^2)} \to \frac{(1-a^2)}{(1+a+a^2)^2} = f'(a)$$

as $x \to a$. Thus f is differentiable at a.

5. *Finish the statement of the Mean Value Theorem. Using just the Mean Value Theorem, show that if* $0 < \alpha \leq 1$ *then for all* $x > 0$, $(1+x)^\alpha \leq 1 + \alpha x$.

Mean Value Theorem. Let $f : [a,b] \to \mathbf{R}$ be a continuous function. If, in addition, f is differentiable on (a,b) then there is a $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$.

Let $f(x) = (1+x)^\alpha$. This function is differentiable for $x > -1$ so continuous on this interval also. For any $x > 0$ we may apply the Mean Value Theorem to f on the subinterval $[0,x]$. There is some $c \in (0,x)$ such that

$$(1+x)^\alpha - 1 = f(x) - f(0) = f'(c)(x-0) = \alpha(1+c)^{\alpha-1}x \leq \alpha x$$

which is the desired inequality. We have used the fact that $1+c > 1$ and $\alpha - 1 \leq 0$ so that $(1+c)^{\alpha-1} \leq 1$.

If $0 < \alpha < 1$ then $(1+c)^{\alpha-1} < 1$ so we get the strict inequality $(1+x)^\alpha < 1 + \alpha x$ whenever $x > 0$.