Math 3210 § 1. **Second Midterm Exam** Name: Solutions

Treibergs March 1, 2017

1. *Consider the subset $E \subset \mathbf{R}$. State the definition: $S = \sup E$ for some $S \in \mathbf{R}$ (same as $S = \operatorname{lub} E$.) Find $\sup E$. Using just the definition, prove that your answer is correct.*

$$E = \left\{ \frac{m-n}{m+n} : m, n \in \mathbb{N} \right\}$$

$S = \sup E$ is defined to be a real number which is (1) an upper bound: $(\forall x \in E)(x \le S)$ and (2) the least of all upper bounds: $(\forall \varepsilon > 0)(\exists x \in E)(S - \varepsilon < x)$.

We claim $S = 1$ is the least upper bound of E. To see that it is an upper bound, every $x \in E$ has the form for some $m, n \in \mathbb{N}$,

$$x = \frac{m-n}{m+n} < \frac{m+n}{m+n} = 1.$$

To see that it is least, choose $\varepsilon > 0$. By the Archimedean Property, there is $m \in \mathbb{N}$ such that $m > \dfrac{2}{\varepsilon}$. Taking this m and $n = 1 \in \mathbb{N}$ we see that the element $x \in E$ given by

$$x = \frac{m-n}{m+n} = \frac{m-1}{m+1} = 1 - \frac{2}{m+1} > 1 - \frac{2}{m} > 1 - \frac{2}{2/\varepsilon} = 1 - \varepsilon.$$

2. *Suppose that $\{a_n\}$ is a real sequence. State the definition: $A \in \mathbf{R}$ is the* limit of the sequence *$A = \lim_{n\to\infty} a_n$. Suppose the sequence converges to a real number $A = \lim\limits_{n\to\infty} a_n$. Show using just the definition of limit (and NOT the Main Limit Theorem) that $\{b_n\}$ also converges, where $b_n = (a_n + 1)^2$.*

$A = \lim_{n\to\infty} a_n$ means that for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|a_n - A| < \varepsilon \qquad \text{whenever } n > N.$$

We argue that $b_n \to (A+1)^2$ as $n \to \infty$. Since $\{a_n\}$ converges, it is bounded, namely, there is $M \in \mathbf{R}$ such that $|a_n| < M$ for all n. Choose $\varepsilon > 0$. By the convergence $a_n \to A$, there is $N \in \mathbf{R}$ be so large that

$$|a_n - A| < \frac{\varepsilon}{M + |A| + 2} \qquad \text{whenever } n > N.$$

Then for the same N, if $n > N$ we have

$$\begin{aligned}
\left|b_n - (A+1)^2\right| &= \left|(a_n+1)^2 - (A+1)^2\right| \\
&= \left|a_n^2 + 2a_n + 1 - (A^2 + 2A + 1)\right| \\
&= \left|a_n^2 - A^2 + 2a_n - 2A\right| \\
&= |(a_n + A)(a_n - A) + 2(a_n - A)| \\
&= |(a_n + A + 2)(a_n - A)| \\
&= |a_n + A + 2|\,|a_n - A| \\
&\le (|a_n| + |A| + 2)\,|a_n - A| \\
&\le (M + |A| + 2)\,|a_n - A| \\
&< (M + |A| + 2)\,\frac{\varepsilon}{M + |A| + 2} = \varepsilon.
\end{aligned}$$

Hence $b_n \to (A+1)^2$ as $n \to \infty$.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) *Suppose the real sequence $\{a_n\}$ is not bounded above. Then $\lim\limits_{n\to\infty} a_n = \infty$.*

   FALSE. The sequence $a_{2n} = n$ and $a_{2n-1} = 1$ is not bounded above but does not tend to infinity. It is not true that given any large number $B > 1$, that there is an N such that $a_n > B$ for all $n > N$ because every other term is one.

   (b) *Let $I_n = [a_n, b_n]$ be closed, bounded and nested intervals: $I_n \supset I_{n+1}$ for all n. Then $\bigcap\limits_{n\in\mathbf{N}} I_n$ consists of a single point.*

   FALSE. Take the intervals $I_n = \left[-\dfrac{1}{n}, 1 + \dfrac{1}{n}\right]$. Then $\bigcap\limits_{n=1}^{\infty} I_n = [0,1]$, not a single point.

   (c) *Suppose that the real sequence $\{a_m\}$ converges to the real number $a = \lim\limits_{n\to\infty} a_n$. If $a < b$ then $a_n < b$ for all but finitely many n.*

   TRUE. Since $a_n \to a$, for $\varepsilon = b - a > 0$ there is $N \in \mathbf{R}$ such that $|a_n - a| < \varepsilon$ whenever $n > N$. For these n,

$$a_n = a + (a_n - a) \le a + |a_n - a| < a + \varepsilon = a + (b - a) = b.$$

   Thus $a_n < b$ may fail only for the finitely many $n \le N$.

4. *Consider the sequence of products*

$$P_n = \left(1 + \frac{1}{2^1}\right)\left(1 + \frac{1}{2^2}\right)\left(1 + \frac{1}{2^3}\right)\cdots\left(1 + \frac{1}{2^n}\right) = \prod_{i=1}^{n}\left(1 + \frac{1}{2^i}\right)$$

Show that for all n, $P_n \le 4\left(1 - \dfrac{1}{2^n}\right)$. Show that the real limit $L = \lim_{n\to\infty} P_n$ exists. $\left(\textit{This defines the infinite product } \prod\limits_{i=1}^{\infty}\left(1 + \dfrac{1}{2^i}\right).\right)$

We prove $P_n \le 4\left(1 - \dfrac{1}{2^n}\right)$ by induction. In the $n = 1$ base case,

$$P_1 = 1 + \frac{1}{2} = \frac{3}{2} \le 2 = 4\left(1 - \frac{1}{2}\right).$$

For the induction case, assume the inequality holds for some $n \in \mathbb{N}$. Then, by the induction hypothesis,

$$\begin{aligned}
P_{n+1} &= \left(1 + \frac{1}{2^{n+1}}\right) P_n \\
&\le \left(1 + \frac{1}{2^{n+1}}\right) 4\left(1 - \frac{1}{2^n}\right) \\
&= 4\left(1 + \frac{1}{2^{n+1}} - \frac{1}{2^n} - \frac{1}{2^{2n+1}}\right) \\
&\le 4\left(1 - \frac{1}{2^{n+1}}\right),
\end{aligned}$$

proving the induction step. Hence, by induction, the inequality holds for ll $n \in \mathbb{N}$.

$\{P_n\}$ is increasing since $P_1 > 0$ and each P_{n+1} is obtained from P_n by multiplying by $1 + 2^{-(n+1)} > 1$. By (a) the sequence is bounded $P_n \le 4$ for all n. Hence by the Monotone Convergence Theorem, the finite limit $L = \lim\limits_{n\to\infty} P_n$ exists. In fact $\frac{3}{2} = P_1 \le L \le 4$.

5. *State the definition:* $\{S_n\}$ *is a Cauchy Sequence. Prove that the finite limit* $L = \lim_{n\to\infty} S_n$ *exists, where the* S_n *is the partial sum*

$$S_n = \sum_{k=1}^{n} \frac{k \sin k}{3^k}.$$

$\{S_n\}$ is a Cauchy Sequence if for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|S_i - S_j| < \varepsilon \qquad \text{whenever } i, j > N.$$

We prove that the partial sum sequence $\{S_n\}$ is a Cauchy Sequence, hence converges. Choose $\varepsilon > 0$. Let $N \in \mathbf{R}$ be so large that $2\left(\frac{2}{3}\right)^N < \varepsilon$. Assume $i, j > N$. If $i = j$ then $|S_i - S - j| = 0 < \varepsilon$. Without loss of generality we may assume $i > j$, otherwise just swap the roles of i and j. Using $|\sin n| \le 1$ and $n \le 2^n$ for all n,

$$\begin{aligned} |S_i - S_j| &= \left| \sum_{k=1}^{i} \frac{k \sin k}{3^k} - \sum_{k=1}^{j} \frac{k \sin k}{3^k} \right| = \left| \sum_{k=j+1}^{i} \frac{k \sin k}{3^k} \right| \le \sum_{k=j+1}^{i} \frac{k |\sin k|}{3^k} \\ &\le \sum_{k=j+1}^{i} \frac{2^k \cdot 1}{3^k} = \frac{\left(\frac{2}{3}\right)^{j+1} - \left(\frac{2}{3}\right)^{i+1}}{1 - \frac{2}{3}} < 2\left(\frac{2}{3}\right)^j < 2\left(\frac{2}{3}\right)^N < \varepsilon. \end{aligned}$$

We have used the formula for a geometric sum with $r = \frac{2}{3}$. If $i > j$,

$$\sum_{k=j+1}^{i} r^k = \frac{r^{j+1} - r^{i+1}}{1 - r}.$$