Math 3220 § 1. **Final Exam** Name: Solutions
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Questions are from Final Exam of December 11, 2007.
(1.) *Determine whether the function* $f : \mathbf{R}^2 \to \mathbf{R}$ *is differentiable at* $(0,0)$*, where*

$$f(x,y) = \begin{cases} \dfrac{xy^5}{x^4+y^4}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Since $f(x,0) = 0$ and $f(0,y) = 0$, we have partial derivatives $f_x(0,0) = 0$ and $f_y(0,0) = 0$ so that if f were differentiable at zero, its differential would be $df(0,0)(h,k) = 0$. We check that the difference quotient vanishes at $(0,0)$. For $(h,k) \neq (0,0)$,

$$\frac{|f(h,k) - f(0,0) - df(0,0)(h,k)|}{\|(h,k)\|} = \frac{\left|\dfrac{hk^5}{h^4+k^4} - 0 - 0\right|}{\sqrt{h^2+k^2}} = \frac{|hk^5|}{(h^4+k^4)\sqrt{h^2+k^2}}$$
$$\leq \frac{(h^2+k^2)(h^4+k^4)}{2(h^4+k^4)\sqrt{h^2+k^2}} \leq \sqrt{h^2+k^2} = \|(h,k)\|$$

which tends to zero as $(k,h) \to (0,0)$. Thus f is differentiable at $(0,0)$. We used $|hk| \leq \frac{1}{2}(h^2+k^2)$ and $k^4 \leq h^4 + k^4$.

(2.) *Let* $K \subseteq \mathbf{R}^n$ *be a compact subset. Suppose* $\mathbf{x}_k \in K$, $k = 1,2,3,\ldots$ *is a sequence of points in* K*. Show that there is a subsequence* $\mathbf{x}_{k_j}$ *that converges in* K *as* $j \to \infty$*.*

Since K is compact, it is bounded. Since $\{\mathbf{x}_k\} \subseteq K \subseteq \mathbf{R}^n$, it is a bounded sequence. By the Bolzano Weirstrass Theorem, every bounded sequence in Euclidean space has a convergent subsequence. Hence there are $k_j \to \infty$ such that $\mathbf{x}_{k_j} \to \mathbf{x}$ as $j \to \infty$ for some $\mathbf{x} \in \mathbf{R}^n$. But since K is compact, it is also closed. But every closed set contains its limit points, thus $\mathbf{x} \in K$.

(3.) *Show that there is a neighborhood* $U \subseteq \mathbf{R}^3$ *of the point* $(1,2,3)$ *and a* $\mathcal{C}^1$ *function* $G : U \to \mathbf{R}^2$ *such that* $G(1,2,3) = (4,5)$ *and* $f(\mathbf{x}, G(\mathbf{x})) = (27,17)$ *for all* $\mathbf{x} \in U$ *where* $f : \mathbf{R}^5 \to \mathbf{R}^2$ *is given by* $f = (f_1, f_2)$ *with*

$$f_1(x,y,z,u,v) = x + yz + uv,$$
$$f_2(x,y,z,u,v) = xu + yv + z.$$

Find $\mathrm{d}G(1,2,3)$*.*

We use the Implicit Function Theorem to solve for $\mathbf{w} = (u,v)$ in terms of $\mathbf{x} = (x,y,z)$ near $(1,2,3,4,5)$. We check the assumptions. First, f is polynomial, hence $\mathcal{C}^1$. Second the differential $d_{\mathbf{w}}(1,2,3,4,5)$ is given by the 2×2 matrix

$$\left.\begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix}\right|_{(\mathbf{x},\mathbf{w})=(1,2,3,4,5)} = \left.\begin{pmatrix} v & u \\ x & y \end{pmatrix}\right|_{(\mathbf{x},\mathbf{w})=(1,2,3,4,5)} = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$$

whose determinant is 6 so $d_{\mathbf{w}}(1,2,3,4,5)$ is invertible. Hence the IFT applies: there is an open neighborhood $V \subseteq \mathbf{R}^5$ of $(1,2,3,4,5)$, an open neighborhood $U \subseteq \mathbf{R}^3$ of $(1,2,3)$ and a function $G \in \mathcal{C}^1(U, \mathbf{R}^2)$ such that $G(1,2,3) = (4,5)$, $f(\mathbf{x}, G(\mathbf{x})) = (27,17)$ for all $\mathbf{x} \in U$ and if $(\mathbf{x}, \mathbf{w}) \in V$ such that $f(\mathbf{x}, \mathbf{z}) = (27,17)$ then $\mathbf{x} \in U$ and $\mathbf{w} = G(\mathbf{x})$.

By differentiating $f(\mathbf{x}, G(\mathbf{x})) = (27, 17)$ we see that $d_{\mathbf{x}} f + d_{\mathbf{w}} f \circ d_{\mathbf{x}} G = 0$ so that

$$\begin{aligned}
d_{\mathbf{x}} G(1,2,3) &= -\left[d_{\mathbf{w}} f(1,2,3,4,5)\right]^{-1} \circ d_{\mathbf{x}} f(1,2,3,4,5) \\
&= -\left[\left.\begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix}\right|_{(\mathbf{x},\mathbf{w})=(1,2,3,4,5)}\right]^{-1} \left.\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix}\right|_{(\mathbf{x},\mathbf{w})=(1,2,3,4,5)} \\
&= -\left[\left.\begin{pmatrix} v & u \\ x & y \end{pmatrix}\right|_{(\mathbf{x},\mathbf{w})=(1,2,3,4,5)}\right]^{-1} \left.\begin{pmatrix} 1 & z & y \\ u & v & 1 \end{pmatrix}\right|_{(\mathbf{x},\mathbf{w})=(1,2,3,4,5)} \\
&= -\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 3 & 2 \\ 4 & 5 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{6} & -\frac{5}{6} \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 4 & 5 & 1 \end{pmatrix} = \begin{pmatrix} \frac{7}{3} & \frac{7}{3} & 0 \\ -\frac{19}{6} & -\frac{11}{3} & -\frac{1}{2} \end{pmatrix}.
\end{aligned}$$

(4.) *Let $f,\ f_n : \mathbf{R}^2 \to \mathbf{R}$ be functions for $n \in \mathbf{N}$. Suppose that for all $\mathbf{x} \in \mathbf{R}^2$, $\lim_{n\to\infty} f_n(\mathbf{x}) = f(\mathbf{x})$. Determine whether the statement is true or false. If true give a brief reason. If false, give a counterexample.*

(a.) Statement. *For every sequence $\mathbf{x}_k \in \mathbf{R}^2$, $k = 1, 2, 3, \ldots$ which converges $\mathbf{x}_k \to \mathbf{x}$ we have $\lim_{k\to\infty} f_k(\mathbf{x}_k) = f(\mathbf{x})$.*

FALSE. For example, let $f_n(\mathbf{x}) = \dfrac{1}{1 + n^2\|x\|^2}$ which tends to $f(\mathbf{x}) = 0$ if $\mathbf{x} \neq 0$ and $f(0) = 1$. Take $\mathbf{x}_n = (\frac{1}{n}, 0)$. Then $f(\mathbf{x}_n) = \frac{1}{2}$ which does not tend to $f(\mathbf{0}) = 1$. The statement would have been true if the convergence had been uniform.

(b.) Statement. *Suppose all $f_k(\mathbf{x}) \in \mathcal{C}^1(\mathbf{R}^2)$. Then f is continuous.*

FALSE. The example in (a.) has $f_n \in \mathcal{C}^1(\mathbf{R}^2)$ since it is the quotient of smooth nonzero functions, but the limit f is not continuous at zero. The statement would have been true if the convergence had been uniform.

(c.) Statement. *Let $R \subseteq \mathbf{R}^2$ be an aligned rectangle. Then $\displaystyle\int_R f(x)\, dV(x) = \lim_{n\to\infty} \int_R f_n(x)\, dV(x)$.*

FALSE. Let $R = [0,1]^2$, $f(x, y) = 0$ and

$$f_n(x,y) = \begin{cases} n^2 x, & \text{if } 0 \le x \le \frac{1}{2n}; \\ n - n^2 x, & \text{if } \frac{1}{2n} < x \le \frac{1}{n}; \\ 0, & \text{otherwise.} \end{cases}$$

Then $f_n \to f$, $\int_R f(\mathbf{x})\, dV(\mathbf{x}) = 0$ but $\int_R f_n(\mathbf{x})\, dV(\mathbf{x}) = \frac{1}{4}$. The statement would have been true if the convergence had been uniform.

(5.) *Let $T = \{(x,y) \in \mathbf{R}^2 : -3 \le x \le 3,\ |x| \le y \le 3\}$. Consider $I = \int_T e^{-y^2}\, dV(x,y)$. Why does the integral I exist? Why can the integral I be reduced to an iterated integral? Evaluate the integral I.*

Since the region T is a triangle in the plane, it is a Jordan region because it is bounded by line segments which have content zero so $V(\partial T) = 0$. Also the function $f(x,y) = \exp(-y^2)$ is continuous over the whole plane, thus on T. Since we have a continuous function on a Jordan region, by the existence theorem over Jordan regions, f is integrable on T.

$T = \{(x,y) : 0 \le y \le 3 \text{ and } \psi(y) \le x \le \phi(y)\ \}$ is a compact Jordan region in the plane determined by the continuous upper and lower functions $\psi(y) \le x \le \varphi(y)$ defined on the Jordan region $B = [0,3]$, where $\psi(y) = -y$ and $\varphi(y) = y$. Since f is continuous in the plane, $f(x,y)$ is integrable on T and integrable with respect to x on the interval $[\psi(y), \varphi(y)]$ for every $y \in [0,3]$. It follows by theorem on iterated integrals over non-rectangular regions determined by an upper and lower function (which follows from Fubini's Theorem) that the integral over T may be written as an iterated integral which reduces the problem to a simple substitution.

$$\begin{aligned}
\int_T f(x,y)\, dV(x,y) &= \int_B \int_{\psi(y)}^{\varphi(y)} f(t,y)\, dt\, dV(y) \\
&= \int_0^3 \int_{-y}^{y} \exp(-y^2)\, dt\, dy \\
&= \int_0^3 2y \exp(-y^2)\, dy \\
&= 1 - e^{-9}.
\end{aligned}$$

(6.) *Let $D \subseteq \mathbf{R}^2$ be the region in the first quadrant bounded by the curves $y = x$, $y^2 - x^2 = 1$, $x^2 + y^2 = 4$, and $x^2 + y^2 = 9$. Find an open set $U \subseteq \mathbf{R}^2$ and a change of variables $\varphi : U \to \mathbf{R}^2$ such that $D = \varphi(R)$, where $R = [0,1] \times [4,9]$, and such that φ is $\mathcal{C}^1$, one-to-one and $\det(\mathrm{d}\varphi(x,y)) \ne 0$ on U. Then find the integral*

$$\int_D \frac{xy}{x^2 + y^2}\, dV(x,y).$$

The first two constraints are equivalent to $y^2 - x^2 = 0$ and $y^2 - x^2 = 1$. Thus we may take

$$\begin{aligned}
s &= y^2 - x^2, \\
t &= y^2 + x^2.
\end{aligned}$$

Solving for (x,y) in terms of (s,t) we find

$$\begin{pmatrix} x \\ y \end{pmatrix} = \varphi \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \sqrt{\dfrac{t-s}{2}} \\ \sqrt{\dfrac{t+s}{2}} \end{pmatrix}$$

where we take the positive square roots. $\varphi \in \mathcal{C}^1(U, \mathbf{R}^2)$ is a one-to-one function if we take the open set $U = \{(s,t) \in \mathbf{R}^2 : t - s > 0 \text{ and } t + s > 0\}$. Note that $\varphi(U)$ is the open first quadrant. Note also that $D = \varphi(R)$ and the rectangle $R \subseteq U$. Since $f(x,y) = \frac{xy}{x^2+y^2}$ is continuous away from $(0,0)$ and φ is $\mathcal{C}^1$ on R, it follows that f is integrable on $\varphi(R)$ and $f(\varphi(s,t))\, |\det(d\varphi(s,t))|$ is integrable on R. The differential is

$$d\varphi(s,t) = \begin{pmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{4}\left(\dfrac{t-s}{2}\right)^{-\frac{1}{2}} & \dfrac{1}{4}\left(\dfrac{t-s}{2}\right)^{-\frac{1}{2}} \\ \dfrac{1}{4}\left(\dfrac{t+s}{2}\right)^{-\frac{1}{2}} & \dfrac{1}{4}\left(\dfrac{t+s}{2}\right)^{-\frac{1}{2}} \end{pmatrix}$$

so that

$$\det(d\varphi(s,t)) = -\frac{1}{8}\left(\frac{t-s}{2}\right)^{-\frac{1}{2}}\left(\frac{t+s}{2}\right)^{-\frac{1}{2}},$$

which is nonzero for all $(s,t) \in U$. Thus the change of variables formula applies to $D \subseteq U$.

$$\begin{aligned}
\int_D \frac{xy}{x^2+y^2}\,dV(x,y) &= \int_{\varphi([0,1]\times[4,9])} f(x,y)\,dV(x,y) \\
&= \int_{[0,1]\times[4,9]} f(\varphi(s,t))\ |\det(d\varphi(s,t))|\,dV(s,t) \\
&= \int_{[0,1]\times[4,9])} \frac{1}{t}\left(\frac{t-s}{2}\right)^{\frac{1}{2}}\left(\frac{t+s}{2}\right)^{\frac{1}{2}}\frac{1}{8}\left(\frac{t-s}{2}\right)^{-\frac{1}{2}}\left(\frac{t+s}{2}\right)^{-\frac{1}{2}}\,dV(s,t) \\
&= \frac{1}{8}\int_0^1\int_4^9 \frac{1}{t}\,dt\,ds \\
&= \frac{1}{8}\log\left(\frac{9}{4}\right).
\end{aligned}$$

(7.) *Let* $f : \mathbf{R}^2 \to \mathbf{R}$ *,* $\psi : \mathbf{R} \to \mathbf{R}$ *be continuous functions such that* $0 \le \psi(x)$.
Show that $g(x) = \int_0^{\psi(x)} f(x,y)\,dV(y)$ *is continuous at all* $x \in \mathbf{R}$.

The idea is to split the integral into two parts where in one the dependence on x is in the upper limit and in the other the dependence is as an argument of f.

Fix $a \in \mathbf{R}$. To show that g is continuous at $a \in \mathbf{R}$, we have to show the definition of continuity is satisfied: that for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|g(x) - g(a)| < \varepsilon \qquad \text{whenever } x \in \mathbf{R} \text{ and } |x-a| < \delta.$$

Let $M = \sup\{\psi(x) : x \in [a-1, a+1]\}$ and $N = \sup\{|f(x,y)| : (x,y) \in [a-1,a+1]\times[-1,M+1]\}$. The suprema exist because continuous functions are bounded on compact sets. Now choose $\varepsilon > 0$. By the continuity of ψ, there is $\delta_1 > 0$ so that

$$|\psi(x) - \psi(a)| < \frac{\varepsilon}{2N+1} \qquad \text{whenever } x \in \mathbf{R} \text{ and } |x-a| < \delta_1.$$

Since f is continuous on the compact set $R = [a-1,a+1]\times[-1,M+1]$, it is uniformly continuous. There is a $\delta_2 > 0$ such that

$$|f(x,y) - f(a,b)| < \frac{\varepsilon}{2M+1} \qquad \text{whenever } (x,y),(a,b) \in R \text{ and } \|(x,y)-(a,b)\| < \delta_2.$$

I claim $\delta = \min\{\delta_1, \delta_2, 1\}$ will work for g. Suppose that $x \in \mathbf{R}$ and $|x-a| < \delta$ so $x \in [a-1, a+1]$. For simplicity let's argue in case $\psi(x) \ge \psi(a)$. Then since $\psi(x), \psi(a) \in [-1, M+1]$, $|f(x,y)| \le N$

and $\|(x,y)-(a,y)\| < \delta$ for $y \in [\psi(a), \psi(x)] \subseteq [-1, M+1]$ so

$$\begin{aligned}
|g(x)-g(a)| &= \left| \int_0^{\psi(x)} f(x,y)\,dy - \int_0^{\psi(a)} f(a,y)\,dy \right| \\
&= \left| \int_0^{\psi(x)} f(x,y) - f(a,y)\,dy + \int_0^{\psi(x)} f(a,y)\,dy - \int_0^{\psi(a)} f(a,y)\,dy \right| \\
&\le \left| \int_0^{\psi(x)} f(x,y) - f(a,y)\,dy \right| + \left| \int_{\psi(a)}^{\psi(x)} f(x,y)\,dy \right| \\
&\le \int_0^{\psi(x)} |f(x,y) - f(a,y)|\,dy + \int_{\psi(a)}^{\psi(x)} |f(x,y)|\,dy \\
&\le \int_0^{\psi(x)} \frac{\varepsilon}{2M+1}\,dy + \int_{\psi(a)}^{\psi(x)} N\,dy \\
&= \frac{\psi(x)\varepsilon}{2M+1} + N|\psi(x) - \psi(a)| \\
&< \frac{M\varepsilon}{2M+1} + \frac{N\varepsilon}{2N+1} \\
&< \varepsilon.
\end{aligned}$$

If $\psi(a) > \psi(x)$ then swap upper and lower limits in the third line and argue similarly. We have shown g is continuous at a, and as a was arbitrary, continuous on $\mathbf{R}$.

(8.) *Define: $E \subseteq \mathbf{R}^n$ is a Jordan Region. Let $f : [a,b] \to \mathbf{R}$ be be a nonnegative integrable function. Show that E is a Jordan region and find its volume* $\mathrm{V}(E)$*, where*

$$E = \{(x,y) \in \mathbf{R}^2 : a \le x \le b, 0 \le y \le f(x)\}.$$

A bounded set $E \subseteq \mathbf{R}^d$ is a *Jordan Region* if its characteristic function χ_E is integrable on some aligned rectangle R containing E, or equivalently, its volume $V(E) = \int_R \chi_E\,dV$ exists.

The idea is that the union of the strips under f over the little subintervals of the partition that shows f is integrable gives an approximation to E so that the integral of f is the area of E.

We are given that $0 \le f$ is an integrable function. Hence it is bounded: there is $M \in \mathbf{R}$ such that $M = \sup\{f(x) : x \in [a,b]\}$. Take the rectangle $R = [a,b] \times [0,M]$ so that $E \subseteq R$. To show that χ_E is integrable on R, we will use the theorem that says that χ_E s integrable on R if and only if for every $\varepsilon > 0$ there is a partition $\mathcal{P}$ of R such that

$$U(\chi_E, \mathcal{P}) - L(\chi_E, \mathcal{P}) < \varepsilon.$$

Since f is integrable on $[a,b]$, there is a partition

$$\mathcal{G} = \{a = x_0 \le x_1 \le x_2 \le \cdots \le x_k = b\}$$

such that one dimensional upper minus lower sums satisfy

$$U(f,\mathcal{G}) - L(f,\mathcal{G}) = \sum_{i=1}^{k} (M_i - m_i)(x_i - x_{i-1}) < \varepsilon$$

where

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x), \qquad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x).$$

Now use the x_i's as cut points of $[a, b]$ in the x-direction and use both sets M_i and m_i as cut points of $[0, M]$ for the y direction. Subdivide the vertical further, so that no rectangle has height greater than ε. We may take the y-cut points

$$\begin{aligned}\{M_i : i = 1, \ldots, k\} \cup \{m_j : j = 1, \ldots, k\} \cup \{\varepsilon h : h \in \mathbb{N} \text{ and } \varepsilon h \leq M\} \\ = \{0 = y_0 \leq y_1 \leq y_2 \leq \cdots \leq y_\ell = M\}.\end{aligned}$$

Together they make a two dimensional partition $\mathcal{P}$ of R.

Denote the subrectangles of $\mathcal{P}$ by $R_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Let

$$M_{i,j} = \sup_{(x,y) \in R_{i,j}} \chi_E(x, y), \qquad m_{i,j} = \inf_{(x,y) \in R_{i,j}} \chi_E(x, y).$$

Observe that $M_{i,j} = 1$ exactly when $R_{i,j} \cap E \neq \emptyset$, which happens for rectangles that touch and are below the graph of f. If f takes its max on $[x_{i-1}, x_i]$ this is for $y_{j-1} \leq M_i$ and if f does not take its max then for $y_j \leq M_i$. Similarly, $m_{i,j} = 1$ exacty when $R_{i,j} \subset E$. If f takes its min on $[x_{i-1}, x_i]$ this is for $y_{j+1} \leq m_i$ and if f does not take its min then for $y_j \leq m_i$.

Consider the upper sum for a fixed x_i.

$$\sum_{j=1}^{\ell} M_{i,j} V(R_{i.j}) \leq \sum_{j: y_{j-1} \leq M_i} (x_i - x_{i-1})(y_j - y_{j-1}) \leq (M_i + y_j - y_{j-1})(x_i - x_{i-1}) \leq (M_i + \varepsilon)(x_i - x_{i-1}).$$

Also the lower sum for a fixed x_i.

$$\sum_{j=1}^{\ell} m_{i,j} V(R_{i.j}) \geq \sum_{j: y_{j+1} \leq m_i} (x_i - x_{i-1})(y_j - y_{j-1}) \geq (m_i - y_{j+1} + y_j)(x_i - x_{i-1}) \geq (m_i - \varepsilon)(x_i - x_{i-1}).$$

It follows that the difference for a fixed x_i is

$$\sum_{j=1}^{\ell} (M_{i,j} - m_{i,j}) V(R_{i.j}) \leq (M_i - m_i + 2\varepsilon)(x_i - x_{i-1}).$$

Summing over i gives the difference for χ_E

$$\begin{aligned}U(\chi_E, \mathcal{P}) - L(\chi_E, \mathcal{P}) &= \sum_{i=1}^{k} \sum_{j=1}^{\ell} (M_{i,j} - m_{i,j}) V(R_{i.j}) \\ &\leq \sum_{i=1}^{k} (M_i - m_i + 2\varepsilon)(x_i - x_{i-1}) \\ &= U(f, \mathcal{G}) - L(f, \mathcal{G}) + 2\varepsilon(b - a) \\ &< [1 + 2(b - a)]\varepsilon.\end{aligned}$$

As ε was arbitrary, this shows that E is a Jordan Domain.

Moreover, the upper and lower sums approximate the integrals. Nomely,

$$\begin{aligned}U(\chi_E, \mathcal{P}) &= \sum_{i=1}^{k} \sum_{j=1}^{\ell} M_{i,j} V(R_{i.j}) \\ &\leq \sum_{i=1}^{k} (M_i + \varepsilon)(x_i - x_{i-1}) \\ &= U(f, \mathcal{G}) + \varepsilon(b - a).\end{aligned}$$

Similarly

$$L(\chi_E, \mathcal{P}) \geq L(f, \mathcal{G}) - \varepsilon(b - a).$$

The upper and lower sums approximate the integral of f:

$$\int_a^b f(x)\, dx - \varepsilon \leq L(f, \mathcal{G}) \leq U(f, \mathcal{G}) \leq \int_a^b f(x)\, dx + \varepsilon$$

Hence

$$\begin{aligned}\int_a^b f(x)\, dx - \varepsilon - \varepsilon(b-a) \leq L(f, \mathcal{G}) - \varepsilon(b-a) \leq \\ L(\chi_E, \mathcal{P}) \leq V(E) = \int_R \chi_E\, dV \leq U(\chi_E, \mathcal{P}) \\ \leq U(f, \mathcal{G}) + \varepsilon(b-a) \leq \int_a^b f(x)\, dx + \varepsilon + \varepsilon(b-a).\end{aligned}$$

Since ε is arbitrary we see that the “area under the graph of f is the integral”

$$V(E) = \int_a^b f(x)\, dx.$$