Math 3220 § 1. **Sample Problems for the Second Midterm Exam** Name: Problems With Solutions
Treibergs $\sigma\tau$ September 28. 2007

Questions 1–10 appeared in my Fall 2000 and Fall 2001 Math 3220 exams.

(1) Let E be a subset of $\mathbf{R}^n$.
  a. Define: E is *open*.
  b. Let $\mathbf{a} \in \mathbf{R}^n$ and $r > 0$. Show that $E = \{\mathbf{x} \in \mathbf{R}^n : \|\mathbf{x} - \mathbf{a}\| > r\}$ is open.

(2) Let $E \in \mathbf{R}^n$.
  a. Define the *closure*, $\overline{E}$.
  b. Show that if $\mathbf{x} \in \overline{E}$ then for every $\varepsilon > 0$, $B_\varepsilon(\mathbf{x}) \cap E \neq \emptyset$. ($B_\varepsilon(\mathbf{x})$ is an open ε-ball about $\mathbf{x}$.)

(3) Suppose $E \subseteq F \subseteq \mathbf{R}^n$. Then the interiors satisfy $E^\circ \subseteq F^\circ$ and that the boundary is contained in the closure $\partial E \subseteq \overline{F}$.

(4) Let $E = [0,1] \cap \mathbf{Q}$, the set of rational points between zero and one. Determine whether the set E is open, closed, or neither. Prove your answer.

(5) Using just the definition of "open set" in $\mathbf{R}^n$, show that $E = \{(x,y) \in \mathbf{R}^2 : x > 0\}$. is an *open set*.

(6) Prove if true, give a counterexample if false:
  a. Let $E \subseteq \mathbf{R}^n$ and $G \subseteq E$ be relatively open. Then for any point $\mathbf{x} \in G$ there is a $\delta > 0$ so that the open δ-ball about $\mathbf{x}$, $B_\delta(\mathbf{x}) \cap E \subseteq G$.
  b. Let $E \in \mathbf{R}^n$ be a set which is *not open*, and suppose $\{\mathbf{x}_n\}$ is a sequence in E which converges $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$ in $\mathbf{R}^n$. Then $\mathbf{x} \in E$.

(7) Prove if true, give a counterexample if false:
  a. Let $E \in \mathbf{R}^n$. If the boundary ∂E is connected then E is connected.
  b. Let $E \subseteq \mathbf{R}^n$. A point is not in the closure $\mathbf{x} \notin \overline{E}$ if and only if there is an open set $\mathcal{O} \subseteq \mathbf{R}^n$ such that $\mathbf{x} \in \mathcal{O}$ but $\mathcal{O} \cap E = \emptyset$.
  c. Let $E \subseteq \mathbf{R}^n$. Then the interior points E° are relatively open in E.

(8) Let $E = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 \le 2\}$. Using only the definition of connectedness, the fact that intervals are the only connected sets connected in $\mathbf{R}^1$, and properties of continuous functions, show that E is a connected subset of $\mathbf{R}^2$.

(9) Prove if true, give a counterexample if false:
  a. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be continuous and $G \subseteq \mathbf{R}^m$ be open. Then for any point $\mathbf{x} \in f^{-1}(G)$ there is a $\delta > 0$ so that the open δ-ball about $\mathbf{x}$, $B_\delta(\mathbf{x}) \subseteq f^{-1}(G)$.
  b. Let $\Omega \subseteq \mathbf{R}^n$ be open and $f : \Omega \to \mathbf{R}^m$ be continuous. Then $f(\Omega)$ is open.
  c. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be continuous and $E \subseteq \mathbf{R}^m$. Suppose E is connected in $\mathbf{R}^m$. Then $f^{-1}(E)$ is connected in $\mathbf{R}^n$.

(10) Let $K \subseteq \mathbf{R}^2$ be a compact set. Suppose $\{\mathbf{x}_n\}_{n\in\mathbf{N}} \subseteq K$ is a sequence in K which is a Cauchy sequence in $\mathbf{R}^2$. Then there is a point $\mathbf{k} \in K$ so that $\mathbf{x}_n \to \mathbf{k}$ as $n \to \infty$.

---

(11) Let L be a linear transformation $L : \mathbf{R}^n \to \mathbf{R}^m$ and let $f(\mathbf{x}) = L\mathbf{x}$. Suppose that $\{\mathbf{x}_k\}$ is a sequence in $\mathbf{R}^n$ that converges $\mathbf{x}_k \to \mathbf{a}$ as $k \to \infty$. Show that $f(\mathbf{x}_k) \to f(\mathbf{a})$ as $k \to \infty$.

(12) Suppose that $\{\mathbf{x}_k\}_{k\in\mathbf{N}} \subseteq \mathbf{R}^3$ is a bounded sequence of points. Show that there is a convergent subsequence. (You may assume the Bolzano-Weierstraß Theorem for $\mathbf{R}^1$ but not for $\mathbf{R}^n$.)

(13) Let $\{\mathbf{x}_k\}_{k\in\mathbf{N}}$ be a sequence in $\mathbf{R}^n$. Prove that $\mathbf{x}_k \to \mathbf{a}$ as $k \to \infty$ if and only if for every open set $G \ni \mathbf{a}$ there is an $N \in \mathbf{N}$ so that for every $k \in \mathbf{N}$, if $k \ge N$ then $\mathbf{x}_k \in G$.

(14) Let $F \in \mathbf{R}^n$ be a set. Show that F is closed if and only if F contains all limits of sequences in F, that is, if $\{\mathbf{x}_k\}_{k\in\mathbf{N}}$ is a sequence in F which converges in $\mathbf{R}^n$, *i.e.*, $\mathbf{x}_k \to \mathbf{a}$ as $k \to \infty$ to some $\mathbf{a} \in \mathbf{R}^n$ then $\mathbf{a} \in F$.

(15) Suppose $S_i \subseteq \mathbf{R}^n$ are closed nonempty sets which are contained in the compact set K. Assume that the subsets form a decreasing sequence $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$. Then they have a nonempty intersection $\bigcap_{i\in\mathbf{N}} S_i \neq \emptyset$.

(16) $E = [0,1] \cap \mathbf{Q}$, the set of rational points between zero and one, is not compact.

(17) *Theorem.* Suppose $E \subseteq \mathbf{R}^n$ is bounded and $f : E \to \mathbf{R}^m$ is uniformly continuous. Then $f(E)$ is bounded. This would not be true if "uniformly continuous" were replaced by "continuous."

(18) *Theorem.* Let $\mathcal{S} = [0,1] \times [0,1] \subseteq \mathbf{R}^2$ and $F : \mathcal{S} \to \mathbf{R}$ be continuous. Then F is not one to one.

Solutions.

(1.) Let E be a subset of $\mathbf{R}^n$. *Definition* : E is *open* if for every point $\mathbf{x} \in E$ there is an $\varepsilon > 0$ so that the open ε-ball about $\mathbf{x}$ is in E, namely $B_\varepsilon(\mathbf{x}) \subseteq E$.

Theorem. Let $\mathbf{a} \in \mathbf{R}^n$ and $r > 0$, then $E = \{\mathbf{x} \in \mathbf{R}^n : \|\mathbf{x} - \mathbf{a}\| > r\}$ is open.

Proof. Choose $\mathbf{y} \in E$. Then $\|\mathbf{y} - \mathbf{a}\| > r$. Let $\varepsilon = \|\mathbf{y} - \mathbf{a}\| - r > 0$. Then I claim that $B_\varepsilon(\mathbf{y}) \subseteq E$ so E is open. To see the claim, choose $\mathbf{z} \in B_\varepsilon(\mathbf{y})$. Then $\|\mathbf{z} - \mathbf{y}\| < \varepsilon$. By the triangle inequality $\|\mathbf{z} - \mathbf{a}\| = \|\mathbf{z} - \mathbf{y} + \mathbf{y} - \mathbf{a}\| \ge \|\mathbf{y} - \mathbf{a}\| - \|\mathbf{z} - \mathbf{y}\| > \|\mathbf{y} - \mathbf{a}\| - \varepsilon = \|\mathbf{y} - \mathbf{a}\| - (\|\mathbf{y} - \mathbf{a}\| - r) = r$, hence, $\mathbf{z} \in E$.

(2.) Let $E \subseteq \mathbf{R}^n$. *Definition* : The *closure* $\overline{E} = \cap\{F : F \subseteq \mathbf{R}^n$ is closed and $E \subseteq F\}$.

Theorem. If $\mathbf{x} \in \overline{E}$ then for every $\varepsilon > 0$ we have $B_\varepsilon(\mathbf{x}) \cap E \ne \emptyset$.

Proof. Suppose it is false for some $\mathbf{x}$. Then there is an $\varepsilon_0 > 0$ and a ball $B_{\varepsilon_0}(\mathbf{x})$ so that $B_{\varepsilon_0}(\mathbf{x}) \cap E = \emptyset$. It follows that $E \subseteq F = \mathbf{R}^n \backslash B_{\varepsilon_0}(\mathbf{x})$ which is a closed set since it is the complement of the open ball, thus it is one of the F's in the intersection definition of closure. Hence, $\overline{E} \subseteq F = \mathbf{R}^n \backslash B_{\varepsilon_0}(\mathbf{x})$. It follows that that $\overline{E} \cap B_{\varepsilon_0}(\mathbf{x}) = \emptyset$ thus $\mathbf{x} \notin \overline{E}$.

(3.) *Theorem.* Suppose $E \subseteq F \subseteq \mathbf{R}^n$. Then the interiors satisfy $E^\circ \subseteq F^\circ$ and that the boundary is contained in the closure $\partial E \subseteq \overline{F}$.

Proof. Recall that the interior is $E^\circ = \cup\{G : G \subseteq \mathbf{R}^n$ is open and $G \subseteq E\}$. Thus if $\mathbf{x} \in E^\circ$ there is an open set $G \subseteq \mathbf{R}^n$ such that $\mathbf{x} \in G \subseteq E$. But $E \subseteq F$ so $G \subseteq F$ is an open set, which is included in the union $F^\circ = \cup\{G : G \subseteq \mathbf{R}^n$ is open and $G \subseteq F\}$. Thus $\mathbf{x} \in G \subseteq F^\circ$.

The closure is defined to be $\overline{F} = \cap\{C : C \subseteq \mathbf{R}^n$ is closed and $F \subseteq C\}$. The boundary is defined to be $\partial E = \overline{E} \backslash E^\circ$ which is contained in $\overline{E}$. Also $\overline{E} = \cap\{H : H \subseteq \mathbf{R}^n$ is closed and $E \subseteq H\}$. If $C \subseteq \mathbf{R}^n$ is any closed set such that $F \subseteq C$ then $E \subseteq F \subseteq C$ so all C's occur as one of the H's in the intersection definition of $\overline{E}$. It follows that $\overline{E} \subseteq \overline{F}$ whence $\partial E \subseteq \overline{E} \subseteq \overline{F}$.

(4.) *Theorem.* Let $E = [0, 1] \cap \mathbf{Q}$, the set of rational points between zero and one. The set E is neither open nor closed.

Proof. To show that E is not open, we show that it is not the case that for every $x \in E$, there exists a $\delta > 0$ so that the ball $B_\delta(x) \subseteq E$. This negation becomes: there is an $x \in E$ so that for every $\delta > 0$, $B_\delta(x)$ is not contained in E, in other words $B_\delta(x) \cap E^c \ne \emptyset$. Take the point $x = 1$ in E. For every $\delta > 0$ there is a number $y \in (1, 1 + \delta) \subseteq B_\delta(1)$. As $y > 1$, so $y \notin E$. Thus for every $\delta > 0$ we have produced $y \in B_\delta(1) \cap E^c$. So E is not open.

A set $E \in \mathbf{R}^n$ is closed if and only if its complement $E^c \subseteq \mathbf{R}^n$ is open. To show that E is not closed, we show that E^c is not open. Choose $z \in E^c$, say $z = \sqrt{2} - 1 \approx .414214\ldots$. By the density of rationals, for every $\delta > 0$ there is a rational number in the interval $q \in B_\delta(z) \cap (0, 1)$. This number $q \in E$, thus, for every $\delta > 0$ there is $q \in B_\delta(z) \cap (E^c)^c$. Thus E^c is not open.

(5.) *Theorem.* Let $E = \{(x, y) \in \mathbf{R}^2 : x > 0\}$. Then E is an *open* set.

E is open if for every $(x, y) \in E$ there is $\varepsilon > 0$ so that $B_\varepsilon(x, y) \subseteq E$. Choose $(x, y) \in E$. Thus $x > 0$. Let $\varepsilon = x$. To show $B_\varepsilon(x, y) \subseteq E$, choose $(u, v) \in B_\varepsilon(x, y)$, thus $\|(x, y) - (u, v)\| < \varepsilon$. Now $u = x - (x - u) \ge x - \|(x, y) - (u, v)\| > x - \varepsilon = x - x = 0$ hence $(u, v) \in E$ thus $B_\varepsilon(x, y) \subseteq E$.

(6a.) *Statement* : Let $E \subseteq \mathbf{R}^n$ and $G \subseteq E$ be relatively open. Then for any point $\mathbf{x} \in G$ there is a $\delta > 0$ so that the open δ-ball about $\mathbf{x}$, $B_\delta(\mathbf{x}) \cap E \subseteq G$. TRUE!

Proof. $G \subseteq E$ relatively open means that there is an open set $\mathcal{O} \subseteq \mathbf{R}^n$ so that $G = \mathcal{O} \cap E$. But if $\mathbf{x} \in G \subseteq \mathcal{O}$ then there is $\delta > 0$ so that $B_\delta(\mathbf{x}) \subseteq \mathcal{O}$ and so $B_\delta(\mathbf{x}) \cap E \subseteq \mathcal{O} \cap E = G$.

(6b.) *Statement* : Let $E \in \mathbf{R}^n$ be a set which is *not open*, and suppose $\{\mathbf{x}_n\}$ is a sequence in E which converges $\lim\limits_{n\to\infty} \mathbf{x}_n = \mathbf{x}$ in $\mathbf{R}^n$. Then $\mathbf{x} \in E$. FALSE!

Let $E = (0, 1] \subseteq \mathbf{R}$. E is not open since $(1 - \varepsilon, 1 + \varepsilon) \not\subseteq E$ all $\varepsilon > 0$. But $x_n = 1/n \in E$ for $n \in \mathbf{N}$, $x_n \to 0$ in $\mathbf{R}$ as $n \to \infty$ but $0 \notin E$.

(7a.) *Statement.* Let $E \in \mathbf{R}^n$. If the boundary ∂E is connected then E is connected. FALSE!

Let $E = \{x \in \mathbf{R} : x \ne 0\}$. Then $\partial E = \{0\}$ which is connected (since it is an interval) but $E = E_1 \cup E_2$ where $E_1 = \{x : x > 0\}$ and $E_2 = \{x : x < 0\}$ which are both open, disjoint and nonempty intervals, therefore separate E into two connected components.

(7b.) *Statement.* Let $E \subseteq \mathbf{R}^n$. A point is not in the closure $\mathbf{x} \notin \overline{E}$ if and only if there is an open set $\mathcal{O} \subseteq \mathbf{R}^n$ such that $\mathbf{x} \in \mathcal{O}$ but $\mathcal{O} \cap E = \emptyset$. TRUE!

The closure is $\overline{E} = \bigcap\{F : F \subseteq \mathbf{R}^n \text{ is closed and } E \subseteq F.\}$. If $\mathbf{x}$ is not in this set then there is a closed set $F \subseteq \mathbf{R}^n$ such that $E \subseteq F$ and $\mathbf{x} \notin F$. Then the complement is open with $\mathbf{x} \in \mathcal{O} = \mathbf{R}^n \backslash F$ and $\mathcal{O} \cap E = \emptyset$ so $\mathcal{O}$ is the desired open set. On the other hand, if there is open $\mathcal{O} \ni \mathbf{x}$ such that $E \cap \mathcal{O} = \emptyset$ then $F = \mathbf{R}^n \backslash \mathcal{O}$ is closed and $E \subseteq F$. Because $\overline{E}$ is defined as the intersection of such F's, it follows that $\overline{E} \subseteq F$. But $\mathbf{x} \notin F$ implies $\mathbf{x} \notin \overline{E}$.

(7c.) *Statement.* Let $E \subseteq \mathbf{R}^n$. Then the interior points E° are relatively open in E. TRUE!

The interior is defined to be $E^\circ = \bigcup\{G : G \in \mathbf{R}^n \text{ is open and } G \subseteq E\}$, thus is the union of open sets so is open in $\mathbf{R}^n$. Also, $E^\circ \subseteq E$ follows. Now E° is relatively open in E if there is an open set $\mathcal{O} \subseteq \mathbf{R}^n$ so that $E^\circ = E \cap \mathcal{O}$. But this follows by setting $\mathcal{O} = E^\circ$ which is an open set in $\mathbf{R}^n$ and because $E^\circ \subseteq E$. Hence $E \cap \mathcal{O} = E \cap E^\circ = E^\circ$.

(8.) Let $E = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \le 2\}$. Using only the definition of connectedness, the fact that intervals are the only connected sets connected in $\mathbf{R}^1$, and properties of continuous functions, show that E is a connected subset of $\mathbf{R}^2$.

The set E is path connected. For example if $x, y \in E$ then $f : [0,1] \to E$ given by $f(t) = (1-t)x + ty$ is a continuous path in E. In fact, for $0 \le t \le 1$ and using the Schwarz Inequality, $\|f(t)\|^2 = (1-t)^2\|x\|^2 + 2t(1-t)x \cdot y + t^2\|y\|^2 \le (1-t)^2\|x\|^2 + 2t(1-t)\|x\|\|y\| + t^2\|y\|^2 = ((1-t)\|x\| + t\|y\|)^2 < (2(1-t) + 2t)^2 = 4$ so $f(t) \in E$. The components of f are polynomial so f is continuous.

Since E is path connected, it is connected. If not there are relatively open sets A_1, A_2 in E so that $A_1 \ne \emptyset$, $A_2 \ne \emptyset$, $A_1 \cap A_2 = \emptyset$ and $E = A_1 \cup A_2$. Choose $x \in A_1$ and $y \in A_2$ and a path $\sigma : [0,1] \to E$ so that $\sigma(0) = x$ and $\sigma(1) = y$. $\sigma^{-1}(A_1)$ and $\sigma^{-1}(A_2)$ are relatively open in $[0,1]$, are disjoint because $A_1 \cap A_2 = \emptyset$ implies $\sigma^{-1}(A_1) \cap \sigma^{-1}(A_2) = \sigma^{-1}(A_1 \cap A_2) = \emptyset$, are nonempty because there are $x \in \sigma^{-1}(A_1)$ and $y \in \sigma^{-1}(A_2)$ and $[0,1] \subseteq \sigma^{-1}(A_1) \cup \sigma^{-1}(A_2) = \sigma^{-1}(A_1 \cup A_2) = \sigma^{-1}(E)$. Thus $\sigma^{-1}(A_1)$ and $\sigma^{-1}(A_2)$ disconnect $[0,1]$, which is a contradiction because $[0,1]$ is connected.

(9.) For each part, determine whether the statement is TRUE or FALSE.

(9a.) *Statement.* Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be continuous and $G \subseteq \mathbf{R}^m$ be open. Then for any point $\mathbf{x} \in f^{-1}(G)$ there is a $\delta > 0$ so that the open δ-ball about $\mathbf{x}$, $B_\delta(\mathbf{x}) \subseteq f^{-1}(G)$.

TRUE! Since G is open, there is $\varepsilon > 0$ so that $B_\varepsilon(f(\mathbf{x})) \subseteq G$. But, since f is continuous, for all positive numbers, such as this $\varepsilon > 0$, there is a $\delta > 0$ so that for all $\mathbf{z} \in \mathbf{R}^n$, if $\|\mathbf{z} - \mathbf{x}\| < \delta$ then $\|f(\mathbf{z}) - f(\mathbf{x})\| < \varepsilon$. We claim that for this $\delta > 0$, $B_\delta(\mathbf{x}) \subseteq f^{-1}(G)$. To see it, choose $\mathbf{z} \in B_\delta(\mathbf{x})$ to show $f(\mathbf{z}) \in G$. But such $\mathbf{z}$ satisfies $\|\mathbf{z} - \mathbf{x}\| < \delta$ so that $\|f(\mathbf{z}) - f(\mathbf{x})\| < \varepsilon$ or in other words, $f(\mathbf{z}) \in B_\varepsilon(f(\mathbf{x})) \subseteq G$.

(9b.) *Statement.* Let $\Omega \subseteq \mathbf{R}^n$ be open and $f : \Omega \to \mathbf{R}^m$ be continuous. Then $f(\Omega)$ is open.

FALSE! Counterexample: the constant function $f(\mathbf{x}) = \mathbf{c}$ is continuous but $f(\Omega) = \{\mathbf{c}\}$ is a singleton set which is not open.

(9c.) *Statement.* Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be continuous and $E \subseteq \mathbf{R}^m$. Suppose E is connected in $\mathbf{R}^m$. Then $f^{-1}(E)$ is connected in $\mathbf{R}^n$.

FALSE! Counterexample: $f(x) = x^2$ is continuous from $\mathbf{R}$ to $\mathbf{R}$ but $f^{-1}([1,4]) = [-2,-1] \cup [1,2]$.

(10.) Let $K \subseteq \mathbf{R}^2$ be a compact set. Suppose $\{\mathbf{x}_n\}_{n \in \mathbf{N}} \subseteq K$ is a sequence in K which is a Cauchy sequence in $\mathbf{R}^2$. Then there is a point $\mathbf{k} \in K$ so that $\mathbf{x}_n \to \mathbf{k}$ as $n \to \infty$.

Since $\{\mathbf{x}_n\}$ is Cauchy, it is convergent in $\mathbf{R}^2$: there is a $\mathbf{k} \in \mathbf{R}^2$ so that $\mathbf{x}_n \to \mathbf{k}$ as $n \to \infty$. But as K is compact it is closed. But a closed set contains its limit points, so $\mathbf{k} \in K$.

(11.) *Theorem.* Let L be a linear transformation $L : \mathbf{R}^n \to \mathbf{R}^m$ and let $f(\mathbf{x}) = L\mathbf{x}$. Suppose that $\{\mathbf{x}_k\}$ is a sequence in $\mathbf{R}^n$ that converges $\mathbf{x}_k \to \mathbf{a}$ as $k \to \infty$. Then $f(\mathbf{x}_k) \to f(\mathbf{a})$ as $k \to \infty$.

Proof. A linear transformation is given by matrix multiplication, thus there is a matrix $A = \{a_{ij}\}$ with $i = 1, \ldots, m$, $j = 1, \ldots, n$ so that if $\mathbf{z} = (z(1), z(2), \ldots, z(n))$ then the i-th component of the value is $f(\mathbf{z})(i) = (A\mathbf{z})(i) = \sum_{j=1}^n a_{ij} z(j)$. In other words, if $\mathbf{a}_i$ denotes the i-th row of A, then the $f(\mathbf{z})(i) = \mathbf{a}_i \cdot \mathbf{z}$. This means that $|f(\mathbf{z})(i)| \le \|\mathbf{a}_i\| \, \|\mathbf{z}\|$ by the Cauchy Schwarz inequality. Hence $\|f(\mathbf{z})\|^2 = \sum_{i=1}^m |f(\mathbf{z})(i)|^2 \le M^2 \, \|\mathbf{z}\|^2$ where $M^2 = \sum_{i=1}^m \|\mathbf{a}_i\|^2$ is a constant depending on L only. To prove that $f(\mathbf{x}_k) \to f(\mathbf{a})$ as $k \to \infty$, we must show that for every $\varepsilon > 0$, there is an $N \in \mathbf{N}$ so that for every $k \ge N$, we have $\|f(\mathbf{x}_k) - f(\mathbf{a})\| < \varepsilon$. Now, choose $\varepsilon > 0$. By the fact that $\mathbf{x}_k$ converges, there is an $N \in \mathbf{N}$ so that if $k \ge N$ then $\|\mathbf{x}_k - \mathbf{a}\| < \varepsilon(1+M)^{-1}$. For this N, if $k \ge N$ then by linearity,

$$\|f(\mathbf{x}_k) - f(\mathbf{a})\| = \|A\mathbf{x}_k - A\mathbf{a}\| = \|A(\mathbf{x}_k - \mathbf{a})\| \le M\|\mathbf{x}_k - \mathbf{a}\| \le \frac{M\varepsilon}{1+M} < \varepsilon.$$

(12.) $Theorem.$ Suppose that $\{\mathbf{x}_k\}_{k\in\mathbf{N}} \subseteq \mathbf{R}^3$ is a bounded sequence of points. Then there it has a convergent subsequence.

$Proof.$ Using the boundedness, that there is $M < \infty$ so that $\|\mathbf{x}_k\| \le M$ for all k, we obtain that the p-th coefficient sequence is bounded because $|\mathbf{x}_k(p)| \le \|\mathbf{x}_k\| \le M$ for all k and p. As the sequence $\{\mathbf{x}_k(1)\}$ is bounded, by the Bolzano-Weierstraß Theorem in $\mathbf{R}^1$, there is a subsequence $k_i \to \infty$ as $i \to \infty$ so that $\mathbf{x}_{k_i}(1) \to \mathbf{a}(1)$ converges to some real number as $i \to \infty$. As the sequence $\{\mathbf{x}_{k_i}(2)\}$ is also bounded, by BW again, there is a subsubsequence $k_{i_j} \to \infty$ as $j \to \infty$ so that $\mathbf{x}_{k_{i_j}}(2) \to \mathbf{a}(2)$ converges as $j \to \infty$. We can repeat this one last time. As the sequence $\{\mathbf{x}_{k_{i_j}}(3)\}$ is also bounded, by BW again, there is a subsubsubsequence $k_{i_{j_\ell}} \to \infty$ as $\ell \to \infty$ so that $\mathbf{x}_{k_{i_{j_\ell}}}(3) \to \mathbf{a}(3)$ converges as $\ell \to \infty$. Since the a subsequence of a convergent sequence is convergent, also $\mathbf{x}_{k_{i_{j_\ell}}}(1) \to \mathbf{a}(1)$ and $\mathbf{x}_{k_{i_{j_\ell}}}(2) \to \mathbf{a}(2)$ as $\ell \to \infty$. Now, using the theorem that a sequence in $\mathbf{R}^3$ converges if and only if all of the sequences of components converge, we get that $\mathbf{x}_{k_{i_{j_\ell}}} \to \mathbf{a}$ in $\mathbf{R}^3$ as $\ell \to \infty$. (Usually, since subscripts of subscripts are frowned upon in typography, we denote subsequences by $k' = k_i$, $k'' = k_{i_j}$ and $k''' = k_{i_{j_\ell}}$ or something similar.)

(13.) $Theorem.$ Let $\{\mathbf{x}_k\}_{k\in\mathbf{N}}$ be a sequence in $\mathbf{R}^n$. $\mathbf{x}_k \to \mathbf{a}$ as $k \to \infty$ if and only if for every open set $G \ni \mathbf{a}$ there is an $N \in \mathbf{N}$ so that for every $k \in \mathbf{N}$, if $k \ge N$ then $\mathbf{x}_k \in G$.

$Proof.$ Assume that $\mathbf{x}_k \to \mathbf{a}$ as $k \to \infty$, namely, for every $\varepsilon > 0$ there is and $N \in \mathbf{N}$ so that for every $k \ge N$ we have $\|\mathbf{x}_k - \mathbf{a}\| < \varepsilon$. Now, choose an open set $G \in \mathbf{R}^n$ which contains $\mathbf{a} \in G$. As G is an open set, there is a $\delta > 0$ so that the δ-ball about $\mathbf{a}$ satisfies $B_\delta(\mathbf{a}) \subseteq G$. Now using $\varepsilon = \delta$ in the statement of convergence, there is an $N \in \mathbf{N}$ so that for every $k \ge N$, $\mathbf{x}_k$ is close to $\mathbf{a}$ so that $\|\mathbf{x}_k - \mathbf{a}\| < \delta$. In other words, $\mathbf{x}_k \in B_\delta(\mathbf{a}) \subseteq G$, as claimed.

To show the other direction, assume that for every open $G \ni \mathbf{a}$, there is $N \in \mathbf{N}$ so that for every $k \ge N$, $\mathbf{x}_k \in G$. Choose $\varepsilon > 0$. Let $G = B_\varepsilon(\mathbf{a})$. As the ball is open, there is an $N \in \mathbf{N}$ so that $k \ge N$ implies $\mathbf{x}_k \in G$. Thus $k \ge N$ implies $\|\mathbf{x}_k - \mathbf{a}\| < \varepsilon$. Hence the definition of $\mathbf{x}_k \to \mathbf{a}$ as $k \to \infty$ is satisfied.

(14.) $Theorem.$ Let $F \in \mathbf{R}^n$ be a set. F is closed if and only if F contains all limits of sequences from F. That is, if $\{\mathbf{x}_k\}_{k\in\mathbf{N}}$ is a sequence in F which converges in $\mathbf{R}^n$, $i.e.$, $\mathbf{x}_k \to \mathbf{a}$ as $k \to \infty$ to some $\mathbf{a} \in \mathbf{R}^n$ then $\mathbf{a} \in F$.

$Proof.$ First we argue that a closed set contains its limit points. Suppose we are given a sequence $\{\mathbf{x}_k\}_{k\in\mathbf{N}}$ in F which converges in $\mathbf{R}^n$, $i.e.$, $\mathbf{x}_k \to \mathbf{a}$ as $k \to \infty$ which means for every $\varepsilon > 0$ there is and $N \in \mathbf{N}$ so that for every $k \ge N$ we have $\|\mathbf{x}_k - \mathbf{a}\| < \varepsilon$. We are to show that $\mathbf{a} \in F$. Suppose that it is not the case. Then $\mathbf{a} \in F^c$, which is an open set. By the definition of F^c being an open set, there is a $\delta > 0$ so that $B_\delta(\mathbf{a}) \subseteq F^c$. This contradicts the assumption that the sequence from F approaches $\mathbf{a}$, for we have shown that there exists a $\delta > 0$ so that for all $N \in \mathbf{N}$ there is a $k \ge N$, say $k = N$, so that $\|\mathbf{x}_k - \mathbf{a}\| \ge \delta$ because $\mathbf{x}_k \notin F^c$.

Next we argue that if a set F contains its limit points, then it must be closed. F is closed if and only if its complement F^c is open. Argue by contrapositive. Suppose that F is not closed so F^c is not open. That is, it is not the case that for every $\mathbf{a} \in F^c$ there exists an $\varepsilon > 0$ so that $B_\varepsilon(\mathbf{a}) \subseteq F^c$. Equivalently, there is an $\mathbf{a} \in F^c$ so that for every $\varepsilon > 0$ there is $\mathbf{x} \in B_\varepsilon(\mathbf{a}) \cap F$. Taking $\varepsilon = 1/k$, there is an $\mathbf{x}_k \in B_{1/k}(\mathbf{a}) \cap F$, which is to say $\|\mathbf{x}_k - \mathbf{a}\| < 1/k$. Thus we have found a sequece $\{\mathbf{x}_k\}$ in F such that $\mathbf{x}_k \to \mathbf{a}$ in $\mathbf{R}^n$ as $k \to \infty$, but $\mathbf{a} \notin F$. In other words, F does not contain one of its limit points.

(15.) $Theorem.$ Suppose $S_i \subseteq \mathbf{R}^n$ are closed nonempty sets which are contained in the compact set K. Assume that the subsets form a decreasing sequence $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$. Then they have a nonempty intersection $\bigcap_{i\in\mathbf{N}} S_i \ne \emptyset$.

$Proof.$ Suppose it is false. Then $\bigcap_{i\in\mathbf{N}} S_i = \emptyset$. Let $U_i = \mathbf{R}^n \backslash S_i$ which are open since S_i are closed. By deMogran's formula, $\cup_i U_i = \cup_i (\mathbf{R}^n \backslash S_i) = \mathbf{R}^n \backslash (\cap_i S_i) = \mathbf{R}^n \backslash \emptyset = \mathbf{R}^n$. Thus $\{U_i\}$ is an open cover of K. Since K is compact, there are finitely many $i_1, i_2, \ldots, i_n$ so that $K \subseteq U_{i_1} \cup \cdots \cup U_{i_n} = (\mathbf{R}^n \backslash S_{i_1}) \cup \cdots \cup (\mathbf{R}^n \backslash S_{i_n}) = \mathbf{R}^n \backslash (S_{i_1} \cap \cdots \cap S_{i_n}) = \mathbf{R}^n \backslash S_p$ where $p = \max\{i_1, \ldots, i_n\}$ since the S_i's are nested. But this says $K \cap S_p = \emptyset$ which contradicts the assumption that S_p is a nonempty subset of K.

(16.) *Theorem.* $E = [0,1] \cap \mathbf{Q}$, the set of rational points between zero and one, is not compact.

Proof. We find an open cover without finite subcover. Let $c = 1/\sqrt{2}$ or any other irrational number $c \in [0,1]$. Consider the sets $U_0 = (c, \infty)$ and $U_i = (-\infty, c - 1/i)$ for $i \in \mathbf{N}$. Then $\mathcal{C} = \{U_i\}_{i=0,1,2,\dots}$ is an open cover. For if $x \in E$, since x is rational, $x \neq c$. If $x > c$ then $x \in U_0$. If $x < c$, by the Archimidean property, there is an $i \in \mathbf{N}$ so that $1/i < c - x$. It follows that $c - 1/i > x$ so $x \in U_i$. On the other hand no finite collection will cover. Indeed, if we choose any finite cover it would have to include U_0 to cover $1 \in E$ and therefore take the form $\{U_0, U_{i_1}, \dots, U_{i_J}\}$ for a finite set of numbers $i_1, \dots, i_J \in \mathbf{N}$. Hence if $K = \max\{i_1, \dots, i_J\}$ then $U_0 \cup U_{i_1} \cup \dots \cup U_{i_J} = (-\infty, c - 1/K) \cup (c, \infty)$. But in the gap $[c - 1/K, c]$ there are rational numbers, by the density of rationals. Thus $E \not\subseteq U_0 \cup U_{i_1} \cup \dots \cup U_{i_J}$. (Of course the easy argument is to observe that E is not closed so can't be compact.)

(17.) *Theorem.* Suppose $E \subseteq \mathbf{R}^n$ is bounded and $f : E \to \mathbf{R}^m$ is uniformly continuous. Then $f(E)$ is bounded. This would not be true if "uniformly continuous" were replaced by "continuous."

Proof. One idea is to divide E into finitely many little pieces so that f doesn't vary very much on any one of them. Then the bound on f is basically the max of bounds at one point for each little piece. f is uniformly continuous if for every $\varepsilon > 0$ there is a $\delta > 0$ so that if $\mathbf{x}, \mathbf{y} \in E$ such that $\|\mathbf{x} - \mathbf{y}\| < \delta$ then $\|f(\mathbf{x}) - f(\mathbf{y})\| < \varepsilon$. Fix an $\varepsilon_0 > 0$ and let uniform continuity give $\delta_0 > 0$. Since E is bounded, there is an $R < \infty$ so that $E \subseteq B_R(\mathbf{0})$. Finitely many $\delta_0/2$ balls are required to cover $B_R(\mathbf{0})$, that is, there are points $\mathbf{x}_i \in \mathbf{R}^n$ so that $B_R(\mathbf{0}) \subseteq \cup_{i=1}^J B_{\delta_0/2}(\mathbf{x}_i)$. This can be accomplished by chopping the ball into small enough cubes and taking $\mathbf{x}_i$'s as the centers of the cubes. *e.g.*, the cube $[-\delta_0/5\sqrt{n}, \delta_0/5\sqrt{n}] \times \cdots \times [-\delta_0/5\sqrt{n}, \delta_0/5\sqrt{n}] \subseteq B_{\delta_0/2}(\mathbf{0})$. Choose points of E in those balls that meet E. Let $\mathcal{I} = \{i \in \{1, \dots, J\} : B_{\delta_0/2}(x_i) \cap E \neq \emptyset\}$ and choose $\mathbf{y}_i \in B_{\delta_0/2}(\mathbf{x}_i) \cap E$ if $i \in \mathcal{I}$. Let $M = \max\{\|f(\mathbf{y}_i)\| : i \in \mathcal{I}\}$ be the largest norm among the points $\mathbf{y}_i$ in E. Then the claim is that $f(E) \subseteq B_{M+\varepsilon_0}(0)$. To see this, choose $\mathbf{z} \in E$. Since E is in the union of little balls, there is an index $j \in \mathcal{I}$ so that $\mathbf{z} \in B_{\delta_0/2}(\mathbf{x}_j)$. Since $\mathbf{y}_j \in B_{\delta_0/2}(\mathbf{x}_j)$ also, it follows that $\|\mathbf{z} - \mathbf{y}_j\| = \|\mathbf{z} - \mathbf{x}_j + \mathbf{x}_j - \mathbf{y}_j\| \le \|\mathbf{z} - \mathbf{x}_j\| + \|\mathbf{x}_j - \mathbf{y}_j\| < \delta_0/2 + \delta_0/2 = \delta_0$. By the uniform continuity, $\|f(\mathbf{y}_j) - f(\mathbf{z})\| < \varepsilon_0$. It follows that $\|f(\mathbf{z})\| = \|f(\mathbf{z}) - f(\mathbf{y}_j) + f(\mathbf{y}_j)\| \le \|f(\mathbf{z}) - f(\mathbf{y}_j)\| + \|f(\mathbf{y}_j)\| < \varepsilon_0 + M$ and we are done.

The result doesn't hold if f is not uniformly continuous. Let $E = B_1(0) \backslash \{0\}$ and $f(\mathbf{x}) = \|\mathbf{x}\|^{-1}$. f is continuous on E but $f(E) = (1, \infty)$ is unbounded.

(18.) *Theorem.* Let $\mathcal{S} = [0,1] \times [0,1] \subseteq \mathbf{R}^2$ and $F : \mathcal{S} \to \mathbf{R}$ be continuous. Then F is not one to one.

Proof. (There are probably many other more imaginative ways to show this.) Consider the circle $\sigma(t) = (\frac{1}{2} + \frac{1}{2}\sin t, \frac{1}{2} + \frac{1}{2}\cos t) \in \mathcal{S}$ as $t \in [0, 2\pi]$. Then $f(t) = F(\sigma(t))$ is a periodic continuous function. If f is constant then $F(\sigma(0)) = F(\sigma(\pi))$ so F is not $1-1$. If f is not constant, since $[0, 2\pi]$ is compact, there are points $\theta_0, \theta_1 \in [0, 2\pi]$ where $f(\theta_0) = \inf\{f(t) : t \in [0, 2\pi]\}$ and $f(\theta_1) = \sup\{f(t) : t \in [0, 2\pi]\}$. Also $f(\theta_0) < f(\theta_1)$. For convenience, suppose $\theta_0 < \theta_1$. The point is that the curves $\sigma((\theta_0, \theta_1))$ and $\sigma((\theta_1, \theta_0 + 2\pi))$ are two opposite arcs of the circle running from the minimum of f on the circle to the maximum. And any intermediate value gets taken on at least once in each arc, thus there are two point where f is equal and F is therefore not $1-1$. More precisely, choose any number $f(\theta_0) < y < f(\theta_1)$. By the intermediate value theorem applied to $f : [\theta_0, \theta_1] \to \mathbf{R}$, there is $\theta_3 \in (\theta_0, \theta_1)$ so that $f(\theta_3) = y$. Also by the intermediate value theorem applied to $f : [\theta_1, \theta_0 + 2\pi] \to \mathbf{R}$, there is $\theta_4 \in (\theta_1, \theta_0 + 2\pi)$ so that $f(\theta_4) = y$. Since $\sigma(\theta_3) \neq \sigma(\theta_4)$ because $0 = \theta_1 - \theta_1 < \theta_4 - \theta_3 < \theta_0 + 2\pi - \theta_0 = 2\pi$, it follows that F is not $1-1$ since $F(\sigma(\theta_3)) = F(\sigma(\theta_4))$. The case $\theta_0 > \theta_1$ is similar.