Math 3220 § 2. **First Midterm Exam** Name: Solutions
Treibergs January 30, 2013

1. Let $\{\mathbf{x}_n\}$ be a sequence in $\mathbb{R}^d$.

   (a) State the definition: $\{\mathbf{x}_n\}$ is a *convergent sequence.*

   For some $\mathbf{x} \in \mathbb{R}^d$, we say that $\mathbf{x} = \lim\limits_{n\to\infty} \mathbf{x}_n$ if for every $\varepsilon > 0$ there is an $N \in \mathbb{R}$ such that $\|\mathbf{x}_n - \mathbf{x}\| < \varepsilon$ whenever $n > N$.

   (b) Suppose that $\mathbf{x} \in \mathbb{R}^d$ is a point such that $r = \|\mathbf{x}\| > 0$ and $\lim\limits_{n\to\infty} \mathbf{x}_n = \mathbf{x}$. Then there is a $K \in \mathbb{R}$ such that $\|\mathbf{x}_n\| > \dfrac{r}{2}$ whenever $n > K$.

   Choose $\varepsilon = \dfrac{r}{2}$. By the definition of convergence, there is $K \in \mathbb{R}$ so that $\|\mathbf{x}_n - \mathbf{x}\| < \varepsilon$ whenever $n > K$. But for such n, by the reverse triangle inequality

$$\|\mathbf{x}_n\| = \|\mathbf{x} - (\mathbf{x} - \mathbf{x}_n)\| \ge \big| \|\mathbf{x}\| - \|\mathbf{x} - \mathbf{x}_n)\| \big| \ge \|\mathbf{x}\| - \|\mathbf{x} - \mathbf{x}_n)\| > r - \varepsilon = \frac{r}{2}.$$

2. (a) Define: E is an open set of $\mathbb{R}^d$.

   $E \subset \mathbb{R}^d$ is open if for every $x \in E$ there is $\varepsilon > 0$ such that the whole open ball

$$B_\varepsilon(x) \subset E.$$

   (b) Let $U, V \subset \mathbb{R}$ be open sets. Show that the product $U \times V$ is an open set in the Euclidean plane, where $U \times V = \big\{(x, y) \in \mathbb{R}^2 : x \in U \text{ and } y \in V\big\}$.

   To show that $U \times V$ is open, we have to show that for every $(u, v) \in U \times V$ there is an $\varepsilon > 0$ such that the open ball $B_\varepsilon\big((u, v)\big) \subset U \times V$. Choose $(u, v) \in U \times V$. Since $u \in U$ an open set, there is $\varepsilon_1 > 0$ such that $(u - \varepsilon_1, u + \varepsilon_1) \subset U$. Also, since $v \in V$ an open set, there is $\varepsilon_2 > 0$ such that $(u - \varepsilon_2, u + \varepsilon_2) \subset V$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\} > 0$. Then $B_\varepsilon\big((u, v)\big) \subset U \times V$. To see it, choose $(p, q) \in B_\varepsilon\big((u, v)\big)$. We have

$$|p - u| \le \sqrt{(p-u)^2 + (q-v)^2} = \|(p, q) - (u, v)\| < \varepsilon \le \varepsilon_1;$$
$$|q - v| \le \sqrt{(p-u)^2 + (q-v)^2} = \|(p, q) - (u, v)\| < \varepsilon \le \varepsilon_2.$$

   It follows that $(p, q) \in (u - \varepsilon_1, u + \varepsilon_1) \times (u - \varepsilon_2, u + \varepsilon_2) \subset U \times V$.
   Hence $B_\varepsilon\big((u, v)\big) \subset U \times V$. □

3. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

   (a) **Statement:** Suppose $E \subset \mathbb{R}^d$ and E does not contain all of its limit points. Then E is open.

   FALSE. For example, let $E = (0, 1] \subset \mathbb{R}$. E is not open but it does not contain its limit points: $\{e^{-n}\} \subset E$ but $\lim_{n\to\infty} e^{-n} = 0 \notin E$.

   (b) **Statement:** Let $A, B \subset \mathbb{R}^d$. Then their closures satisfy $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

   FALSE. For example, let $A = (-\infty, 0) \subset \mathbb{R}$ and $B = (0, \infty) \subset \mathbb{R}$. Then $A \cap B = \emptyset$ so $\overline{A \cap B} = \emptyset$ but $\overline{A} = (-\infty, 0]$ and $\overline{B} = [0, \infty)$ so $\overline{A} \cap \overline{B} = \{0\}$.

   (c) **Statement:** The function $|||\,(u, v)\,||| = \left(\sqrt{|u|} + \sqrt{|v|}\right)^2$ provides another norm for $\mathbf{R}^2$.

   FALSE. The triangle inequality fails. For example, let $\mathbf{u} = (1, 4)$ and $\mathbf{v} = (4, 1)$. Then $|||\,(1, 4)\,||| = |||\,(4, 1)\,||| = (\sqrt{1} + \sqrt{4})^2 = 9$. However $|||\,\mathbf{u} + \mathbf{v}\,||| = |||\,(5, 5)\,||| = (\sqrt{5} + \sqrt{5})^2 = 20$ is not less than or equal to $|||\,\mathbf{u}\,||| + |||\,\mathbf{v}\,||| = 9 + 9 = 18$.

4. Let $E \subset \mathbb{R}^d$ be a bounded infinite set in Euclidean space. Then there is a point $z \in \mathbb{R}^d$ such that every open neighborhood of z contains infinitely many points of E. [Note: E may be uncountable so that it may not be given as points of a sequence.]

   We shall find a candidate $z \in \mathbb{R}^d$ using bisection. Then argue that z has the desired property.

   Because E is bounded, for s sufficiently large, E is contained in the cube $Q_1 = [-s, s]^d$. For $j = 1, \ldots, 2^d$, let Q_1^j denote the different closed sub-cubes obtained by cutting Q_1 by the middle coordinate hyperplanes. At least one of $Q_1^j \cap E$ is infinite. If not, $E = \cup_{i=1}^{2^d} Q_1^j \cap E$, a finite union of finite sets. Let Q_2 be one of the Q_1^j's such that $Q_1^j \cap E$ is infinite. Proceeding in this way, at each step we subdivide the cube Q_k in turn into 2^d sub-cubes and let Q_{k+1} be one of these sub-cubes that meets E in infinitely many points.

   The cubes are a nested sequence of nonempty closed and bounded sets

$$Q_1, \supset Q_2 \supset Q_3 \supset Q_4 \supset \cdots$$

   By the nested intervals theorem in $\mathbb{R}^d$, there is a point $z \in \cap_{i=1}^{\infty} Q_i$.

   We end the proof by showing that z is the desired point. Let U be an open neighborhood of z. Hence U is an open set. Thus there is an $\varepsilon > 0$ such that the ball $B_\varepsilon(z) \subset U$. Since the distance between any pair of points in Q_n is at most $2^{2-n} s\sqrt{d}$, for n large enough we have $Q_n \cap E \subset B_\varepsilon(z) \subset U$. But by construction, $Q_n \cap E$ contains infinitely many points of E. □

5. Let $K \subset \mathbb{R}^d$.

   (a) State the definition: *K is a compact set.*

   $K \subset \mathbb{R}^d$ is compact if every open cover of K has a finite subcover.

   (b) Let $E = \left\{ \left(\frac{1}{n}, 0 \right) \in \mathbb{R}^2 : n \in \mathbb{N} \right\}$. Find an open cover of E that does not have a finite subcover. (You do not need to prove that your cover has this property.)

   Let $U_n = \left\{ (x, y) \in \mathbb{R}^2 : x > \dfrac{1}{n+1} \text{ and } y \in \mathbb{R}. \right\}$. The desired cover is $\{U_n\}_{n \in \mathbb{N}}$.

   To see that it is a cover, for any $n \in \mathbb{N}$, $\left(\frac{1}{n}, 0 \right) \in U_n$ so $E \subset \cup_{i=1}^{\infty} U_i$. However, this cover does not have a finite subcover. For any finite subset $\{i_1, \ldots, i_p\} \subset \mathbb{N}$, the corresponding sets $\cup_{j=1}^{p} U_{i_j}$ fail to cover because they don't include the points $\left(\frac{1}{N}, 0 \right)$ when $N > \max\{i_1, \ldots, i_p\}$.

   (c) Let $F = E \cup \{(0,0)\}$. Show that every open cover of F has a finite subcover.

   Let $\{G_a\}_{a \in A}$ be an open cover so all G_a are open and $F \subset \cup_{a \in A} G_a$. Thus one of the sets of this cover, say for $a_0 \in A$, contains $(0, 0) \in G_{a_0}$. As G_{a_0} is open, there is a $\varepsilon > 0$ such that $B_\varepsilon\big((0, 0)\big) \subset G_{a_0}$. There is $K \in \mathbb{N}$ such that $\dfrac{1}{K} \le \varepsilon$. It follows that all points $\left(\frac{1}{n}, 0 \right) \in B_\varepsilon\big((0, 0)\big) \subset G_{a_0}$ whenever $n > K$. For the remaining uncovered points $k = 1, 2, \ldots, K$, we choose sets, corresponding to $a_k \in A$ such that $\left(\frac{1}{k}, 0 \right) \in G_{a_k}$. It follows that finite subcollection

$$\{G_{a_0}, G_{a_1}, \ldots, G_{a_K}\}$$

   is a cover of F: $F \subset G_{a_0} \cup G_{a_1} \cup \cdots \cup G_{a_K}$. □