Math 3220 § 1. **Final Exam** Name:_______________

Treibergs $\sigma\tau$ December 11, 2007

This is a closed book exam except that you are allowed a "cheat sheet," a single 8.5"×11" page of notes. Other notes, books, laptops and text messaging devices are prohibited. Calculators are permitted but are unnecessary. Give complete solutions. Be clear about the order of logic and state the theorems and definitions that you use. There are [135] total points. **Do SEVEN of eight problems.** If you do more than seven problems, only the first seven will be graded. Cross out the problems you don't wish to be graded.

1.	/19
2.	/19
3.	/19
4.	/19
5.	/19
6.	/20
7.	/20
8.	/20
Total	/135

1. [19] Determine whether the following function $f : \mathbf{R}^2 \to \mathbf{R}$ is differentiable at $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{xy^5}{x^4+y^4}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Differentiable at $(0,0)$: ◯	Not differentiable at $(0,0)$: ◯

Math 3220 § 1. **Final Exam** Name:____________
Treibergs December 11, 2007

Your grades will be posted at my office according to | *Secret Id.* :

---

2. [19] Let $K \subseteq \mathbf{R}^n$ be a compact subset. Suppose $\mathbf{x}_k \in K$, $k = 1, 2, 3, \ldots$ is a sequence of points in K. Show that there is a subsequence $\mathbf{x}_{k_j}$ that converges in K as $j \to \infty$.

3. [19] Show that there is a neighborhood $U \subseteq \mathbf{R}^3$ of the point $(1,2,3)$ and a C^1 function $G : U \to \mathbf{R}^2$ such that $G(1,2,3) = (4,5)$ and $f(\mathbf{x}, G(\mathbf{x})) = (27,17)$ for all $\mathbf{x} \in U$ where $f : \mathbf{R}^5 \to \mathbf{R}^2$ is given by $f = (f_1, f_2)$ with

$$f_1(x,y,z,u,v) = x + yz + uv,$$
$$f_2(x,y,z,u,v) = xu + yv + z.$$

Find $\mathrm{d}G(1,2,3)$.

4. Let $f,\ f_n : \mathbf{R}^2 \to \mathbf{R}$ be functions for $n \in \mathbf{N}$. Suppose that for all $x \in \mathbf{R}^2$, $\lim_{n\to\infty} f_n(x) = f(x)$. Determine whether the statement is true or false. If true give a brief reason. If false, give a counterexample.

   (a) [6] **Statement.** For every sequence $\mathbf{x}_k \in \mathbf{R}^2$, $k = 1, 2, 3, \ldots$ which converges $\mathbf{x}_k \to \mathbf{x}$ we have $\lim_{k\to\infty} f_k(\mathbf{x}_k) = f(\mathbf{x})$.

   True ◯ False ◯

   (b) [6] **Statement.** Suppose all $f_k(\mathbf{x}) \in C^1(\mathbf{R}^2)$. Then f is continuous.

   True ◯ False ◯

   (c) [7] **Statement.** Let $R \subseteq \mathbf{R}^2$ be an aligned rectangle. Then $\displaystyle\int_R f(x)\,dV(x) = \lim_{n\to\infty} \int_R f_n(x)\,dV(x)$.

   True ◯ False ◯

5. Let $T = \{(x, y) \in \mathbf{R}^2 : -3 \le x \le 3,\ |x| \le y \le 3\}$. Consider $I = \int_T e^{-y^2}\, dV(x, y)$.

   (a) [5] Why does the integral I exist?

   (b) [5] Why can the integral I be reduced to an iterated integral?

   (c) [9] Evaluate the integral I.

6. Let $D \subseteq \mathbf{R}^2$ be the region in the first quadrant bounded by the curves $y = x$, $y^2 - x^2 = 1$, $x^2 + y^2 = 4$, and $x^2 + y^2 = 9$.

   (a) [10] Find an open set $U \subseteq \mathbf{R}^2$ with $A \subseteq U$, where $A = [0,1] \times [4,9]$ and a function $\phi : U \to \mathbf{R}^2$ such that $\phi(A) = D$, ϕ is C^1, one-to-one and $\det(\mathrm{d}\phi(x,y)) \neq 0$ on U.

   (b) [10] Using (a.), find the integral $\displaystyle\int_D \frac{xy}{x^2+y^2}\, dV(x,y)$.

7. [20] Let $f : \mathbf{R}^2 \to \mathbf{R}$, $\psi : \mathbf{R} \to \mathbf{R}$ be continuous functions such that $0 \le \psi(x)$.
   Show that $g(x) = \displaystyle\int_0^{\psi(x)} f(x,y)\, dV(y)$ is continuous at all $x \in \mathbf{R}$.

8. (a) [3] Define: $E \subseteq \mathbf{R}^n$ is a *Jordan Region.*

(b) [17] Let $f : [a,b] \to \mathbf{R}$ be be a nonnegative integrable function. Show that

$$E = \{(x,y) \in \mathbf{R}^2 : a \le x \le b, 0 \le y \le f(x)\}.$$

is a Jordan region. Find its volume $\mathrm{V}(E)$.