Homework for Math 4530 §1, Spring 2008

Andrejs Treibergs, Instructor

April 18, 2008

Please read the relevant sections in the text *Elementary Differential Geometry* by Andrew Pressley, Springer, 2001. You may find solved exercises in the text similar to the given problems.

Please hand in problems 1a. and 1b. on Monday, Jan. 14.

1. **Parametrizations of curves.**

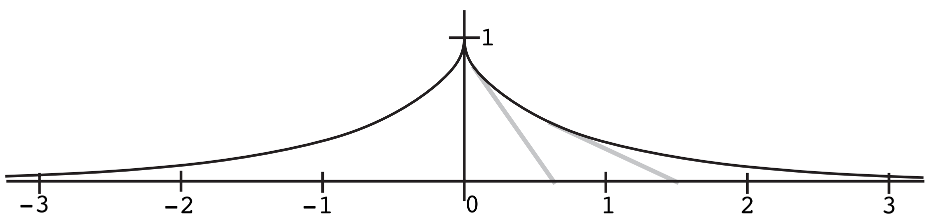

Figure 1: Tractrix

(a) **Tractrix.** Suppose that a Central Park matron is walking her dog. She starts at the origin and walks along the positive x-axis at unit speed. The dog refuses to move and is dragged by a leash of length one. Suppose that the dog starts at $(0, 1)$. Determine the trajectory of the dog, which is the tractrix curve. This curve is sketched in Figure 6 and is is animated on the class homepage

`http : //www.math.utah.edu/ treiberg/M4531.html`

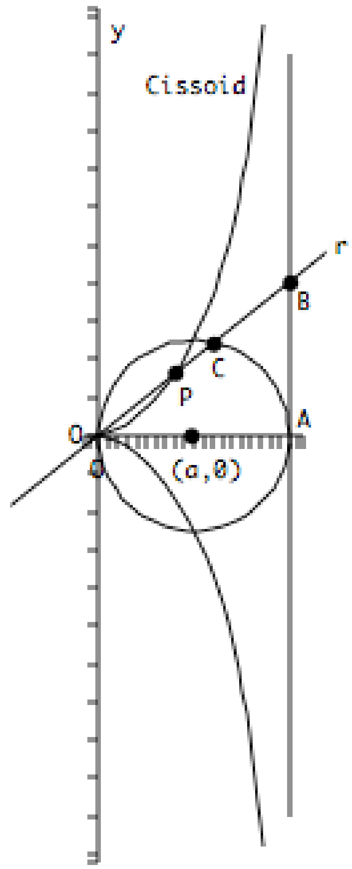

Figure 2: Cissoid of Diocles

(b) **Cissoid of Diocles.** Let OA be the diameter of a circle of radius a, which we suppose to be on the x-axis. Then the y-axis and the line $x = 2a$ are tangent to the circle at the points O and A, resp. A line r is drawn through the origin is allowed to rotate. r meets the circle at a point C and meets the line $x = 2a$ at the point B. Let P be a point on the line r between O and B such that the distance $d(O, P) = d(C, B)$. The trace of the point P as we rotate the line r is a curve called the *Cissoid of Diocles.* The curve is sketched in Figure 2.

i. Prove that a parameterization of the cissoid is given by $\gamma : \mathbb{R} \to \mathbb{R}^2$ where

$$\gamma(t) = \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2} \right).$$

($t = \tan\theta$ where θ is the angle between the x-axis and r.)

ii. Find the Cartesian Equation for the Cissoid of Diocles.

Please hand in **any three** of the problems 2, 3, 4 or 5 on Wed., Jan. 23.

2. **Approximation by zig-zag curves.** Suppose that $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ is a smooth parameterized curve and that $[a, b] \in (\alpha, \beta)$ is a finite interval. The length of the curve restricted to $[a, b]$ is given by
$$L\left(\gamma\big|_{[a,b]}\right) = \int_a^b \|\dot\gamma(t)\|\,dt.$$
Suppose $\mathcal{P}$ is a partition of $[a, b]$ given by finitely many points
$$\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}.$$
Consider the zig-zag approximation to γ given by joining line segments to from $\gamma(t_0)$ to $\gamma(t_1)$ to $\gamma(t_2)$ and so on ending at the linear segment from $\gamma(t_{n-1})$ to $\gamma(t_n)$. The length of the zig-zag approximation is
$$L(\gamma, \mathcal{P}) = \sum_{k=1}^{n} \|\gamma(t_k) - \gamma(t_{k-1})\|.$$
Show that for every $\varepsilon > 0$, there is a $\delta > 0$ so that whenever $\mathcal{P}$ is a partition of $[a, b]$ such that $\|\mathcal{P}\| < \delta$ then
$$\left|L\left(\gamma\big|_{[a,b]}\right) - L(\gamma, \mathcal{P})\right| < \varepsilon.$$
Here, the fineness of the partition is determined by its *mesh size* given by
$$\|\mathcal{P}\| = \max_{t_i, t_{i-1} \in \mathcal{P}} |t_i - t_{i-1}|.$$

3. **Elliptical Helix.** Find the tangent, normal and binormal vectors as well as the curvature and torsion for the *elliptical helix* $\gamma : \mathbb{R} \to \mathbb{R}^3$ given for constants a, b, c, d by
$$\gamma(t) = (at, b\cos dt, c\sin dt).$$

4. **Signed curvature in the plane.** Suppose $f(t) : (\alpha, \beta) \to \mathbb{R}$ is a smooth positive function. Find the signed curvature κ_s of the curve in polar coordinates $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ given by
$$\gamma(t) = (\, f(t)\, \cos t,\; f(t)\, \sin t\,).$$

5. **Curvature of space curves.** Show that there is a closed smooth, regular parameterized curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ such that both the curvature and the torsion functions are positive: $\kappa > 0$ and $\tau > 0$. (Such a curve may be called *dextrose.* A curve is closed if it is L-periodic on $\mathbb{R}$.)

   Suppose that $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ is a regular smooth parameterized curve such that the curvature is positive $\kappa > 0$, and its derivative and the torsion are all nonzero $\dot\kappa = \frac{d\kappa}{ds} \neq 0$ and $\tau \neq 0$. Show that there is a sphere $\mathbf{S}^2 \subset \mathbb{R}^3$ such that $\gamma(t) \in \mathbf{S}^2$ for all $t \in (\alpha, \beta)$ if and only if
$$\frac{1}{\kappa^2} + \frac{(\dot\kappa)^2}{\tau^2\kappa^4} = \text{const.}$$

Please hand in **any two** of the problems 6, 7a or 7b on Monday, Jan. 28.

6. **Rigid motion.** Suppose that $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a smooth, simple closed curve parameterized by arclength. γ is periodic of period L, which is the length of the curve. Show that the length and the enclosed area are preserved when γ is mapped by a rigid motion. A rigid motion is

$$\tilde{\gamma}(s) = T_v \circ R_\theta\big(\gamma(s)\big)$$

where $T_{\mathbf{v}}(\mathbf{x}) = \mathbf{v} + \mathbf{x}$ is translation by a vector $\mathbf{v}$ and R_θ, a rotation by angle θ,

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

[See problem 3.1 of the text.]

7. **Plane Curves.**

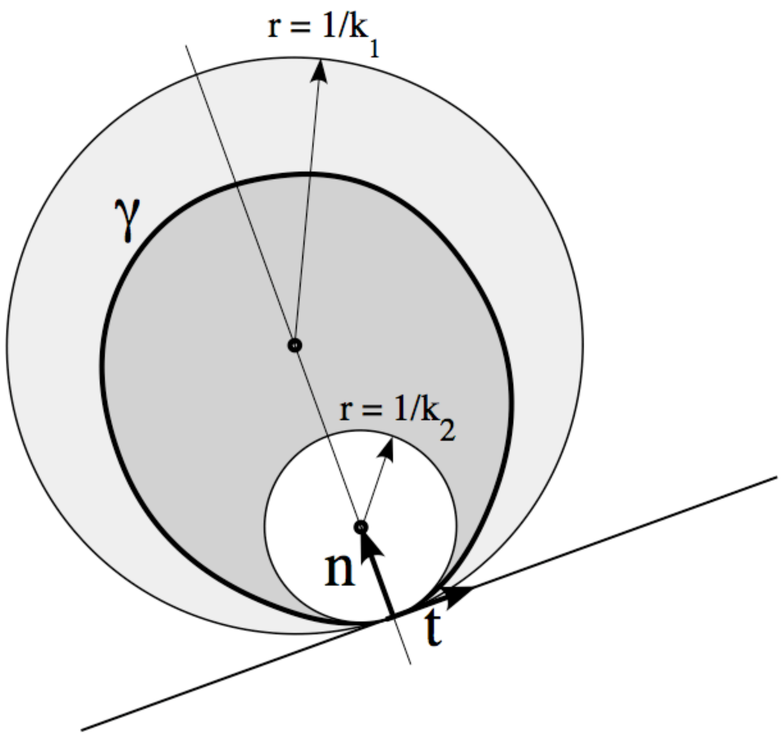

Figure 3: Tangent disks inside and outside of γ.

(a) **A. Schur & E. Schmidt's Theorem.** Suppose that $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a regular, simple closed curve which is positively oriented. Suppose that there are two numbers $0 < k_1 \le k_2$ such that the curvature of the curve satisfies $k_1 \le \kappa(s) \le k_2$ for all s. Show that at each point $\gamma(s_0)$, γ is contained in the tangent disk of radius $1/k_1$ and contains the tangent disk of radius $1/k_2$. (The disk of radius r tangent to γ at $\gamma(s_0)$ has center $\gamma(s_0) + r\mathbf{n}(s_0)$, where $\mathbf{n}(s)$ is the inward-pointing unit normal vector to the curve.)

(b) **Winding number.** Suppose that $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a regular, simple closed curve which is periodic of period L. Let P_0 be a point not on the curve which is *outside* of γ. Show that the winding number

$$w_{\gamma, P_0} = 0.$$

The winding number is the index of the map $\varphi : \mathbb{S}^1_L \to \mathbb{S}^1$ given by

$$\varphi(t) = \frac{\gamma(t) - P_0}{\|\gamma(t) - P_0\|}.$$

[You may assume the Jordan Curve Theorem.]

Please hand in **any two** of the problems 8a, 8b or 9 on Monday, Feb. 4.

8. **Convex Curves.**

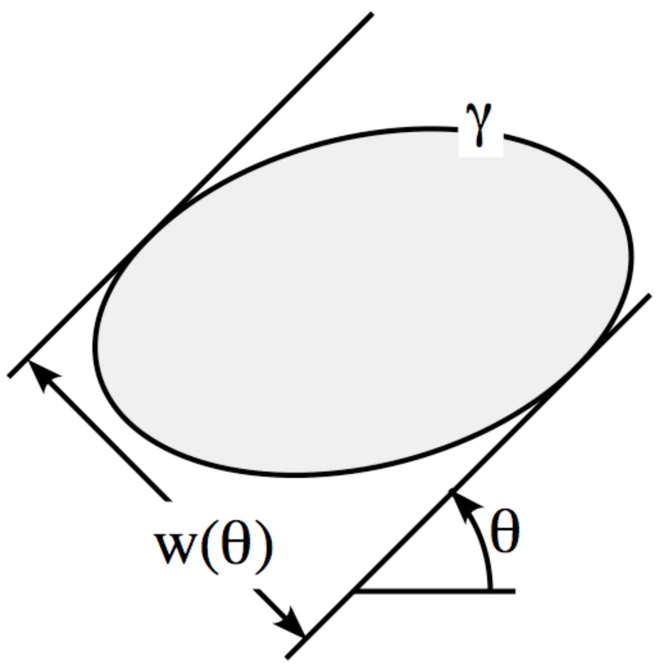

Figure 4: Width of γ in the θ direction.

(a) **Figures of Constant Width.** Suppose that $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a regular, simple closed curve which is positively oriented and has curvature $\kappa > 0$ so that γ is a convex. Such curves are called *ovaloids.* For each direction $\theta \in [0, 2\pi)$, the distance between the two opposite tangent lines in the θ direction is the width of the curve $w(\theta)$. A curve is said to have constant width if there is a constant w_0 such that if $w(\theta) = w_0$ for all θ.

Besides the circle, another constant width curve is the Reuleaux Triangle, drawn by three arcs of a compass [see Fig.5].

i. Find another ovaloid of constant width besides the circle and the Reuleaux Triangle. [Hint: there are lots of them!]

ii. Show that for any ovaloid of constant width w_0, the length is given by $L = \pi w_0$.

(b) **Support Function.** Suppose that $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a regular, simple closed curve which is positively oriented and has curvature $\kappa > 0$. Suppose that 0 is in the interior of γ.

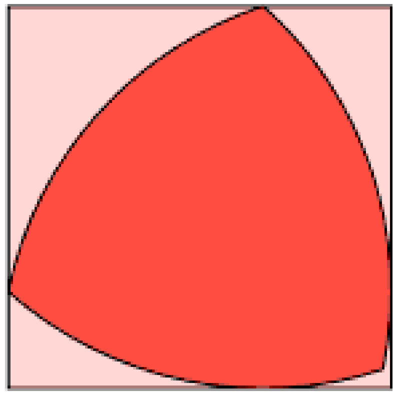

Figure 5: Reuleaux Triangle has constant width..

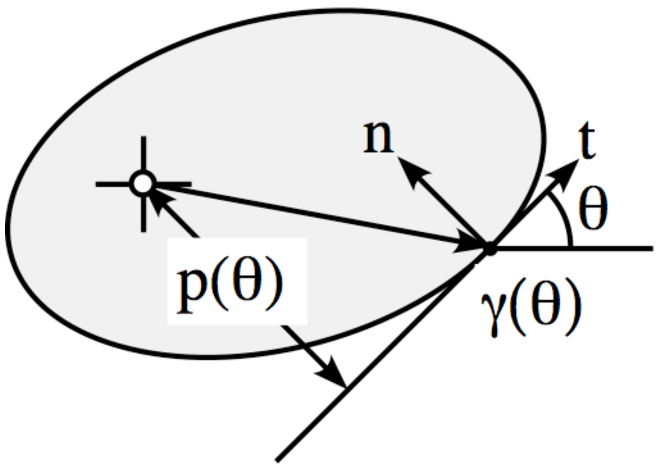

Figure 6: $p(\theta)$ is the distance from the origin to the tangent line in the θ direction.

Since the curve has positive curvature, one can use the angle as a parameterization $\gamma(\theta) : [0, 2\pi) \to \mathbb{R}^2$. The *support* or *pedal* function is the distance between the tangent line in the θ direction and the origin. Thus it satisfies

$$p(\theta) = -\mathbf{n}(\theta) \bullet \gamma(\theta),$$

where $\mathbf{n}(\theta) = (-\sin\theta, \cos\theta)$ is the inner unit normal.

i. Find a formula for $\gamma(\theta)$ in terms of $p(\theta)$ and $\dfrac{dp}{d\theta}(\theta)$

ii. Show that the signed curvature in terms of θ derivatives is

$$\kappa = \frac{1}{p + \dfrac{d^2p}{d\,\theta^2}}.$$

iii. Show that the length is

$$L = \int_0^{2\pi} p(\theta)\, d\theta.$$

iv. Show that the enclosed area is given by

$$A = \frac{1}{2}\int_0^{2\pi} p(\theta)^2 - \left(\frac{dp}{d\theta}\right)^2 d\theta.$$

v. Show that if the ovaloid is contained in a circular disk of radius R, then

$$A \le \frac{1}{2} L R.$$

[Hint: Hurwitz's first proof of the isoperimetric inequality was an application of Wirtinger's inequality to the area formula iv.]

9. **Regular surfaces.** Show that the surface of revolution $\mathcal{S}$ is a regular surface. Suppose $f : (\alpha, \beta) \to \mathbb{R}$ be a smooth function such that $f(z) > 0$ for all $z \in (\alpha, \beta)$. Imagine that the curve $(f(z), 0, z)$ as $z \in (\alpha, \beta)$ is traced out on the $y = 0$ plane and rotated around the z-axis. The resulting surface is the surface of revolution. Alternatively, it is the surface

$$\mathcal{S} = \left\{ (x, y, z) \in \mathbb{R}^3 : \alpha < z < \beta,\ x^2 + y^2 = (f(z))^2 \right\}.$$

Please hand in any two of the Problems 10, 11a or 11b on Wednesday, Feb. 20.

10. **Reparametrizations.**[See problem 4.8 of Pressley.] Let $P_0 = (x_0, y_0, z_0)$ where $z_0 > 1$ be any point of the upper nappe of the hyperboloid with two sheets

$$\mathcal{S} = \{(x, y, z) : -x^2 - y^2 + z^2 = 1\}.$$

Show that a coordinate chart can be given in the neighborhood of P_0 of the form $\sigma : U \to W \cap \mathcal{S}$ where $U \subset \mathbb{R}^2$ and $W \subset \mathbb{R}^3$ are appropriate open sets and

$$\sigma(\rho, \theta) = (\sinh\rho\, \cos\theta,\ \sinh\rho\, \sin\theta,\ \cosh\rho).$$

Use Exercise 4.6 to find another parametrization $\tilde{\sigma} : \tilde{U} \to W \cap \mathcal{S}$ of the same part, and verify that $\tilde{\sigma}$ is a reparametrization of σ.

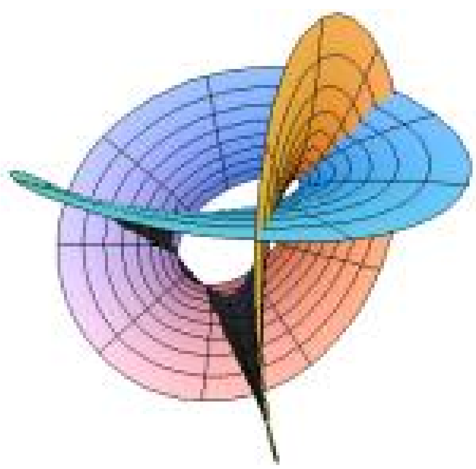

Figure 7: Henneberg Minimal Surface.

11. **Unit normal to surface**

(a) **Tangent Developable Surface.** Let $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ be a smooth regular curve such that $\kappa > 0$. Let $\sigma : U \to \mathbb{R}^3$ be a map given by

$$\sigma(u, v) = \gamma(u) + v\,\mathbf{t}(u)$$

where $\mathbf{t}$ is the unit tangent vector to γ and $U = (\alpha, \beta) \times (0, \infty)$. Show that *tangent developable* $\mathcal{S} = \sigma(U)$ is a regular surface at least near any point $P = \sigma(u_0, v_0)$. Find the unit normal vector and the equation of the tangent plane to $\mathcal{S}$ at P.

(b) **Henneberg Surface.** The Henneberg surface, Fig. 7, is a minimal surface given by the formula $\mathcal{S} = \sigma(U)$, where

$$\sigma(u,v) = \begin{pmatrix} \sinh u \, \cos v - \frac{1}{3} \sinh 3u \, \cos 3v \\ \\ \sinh u \, \sin v + \frac{1}{3} \sinh 3u \, \sin 3v \\ \\ \cosh 2u \, \cos 2v \end{pmatrix},$$

and $U = \mathbb{R}^2 - \{(0, \frac{\pi n}{2}) : n \in \mathbb{Z}\}$. σ is periodic so $\sigma : U \to \mathcal{S}$ is multiply covered, as in the usual polar coordinates of a sphere. Check that the Henneberg surface is, at least locally, a regular surface. Show that the Henneberg surface is not oriented. [Hint: You may first wish to show that $\sigma(u,v) = \sigma(-u, v+\pi)$.]

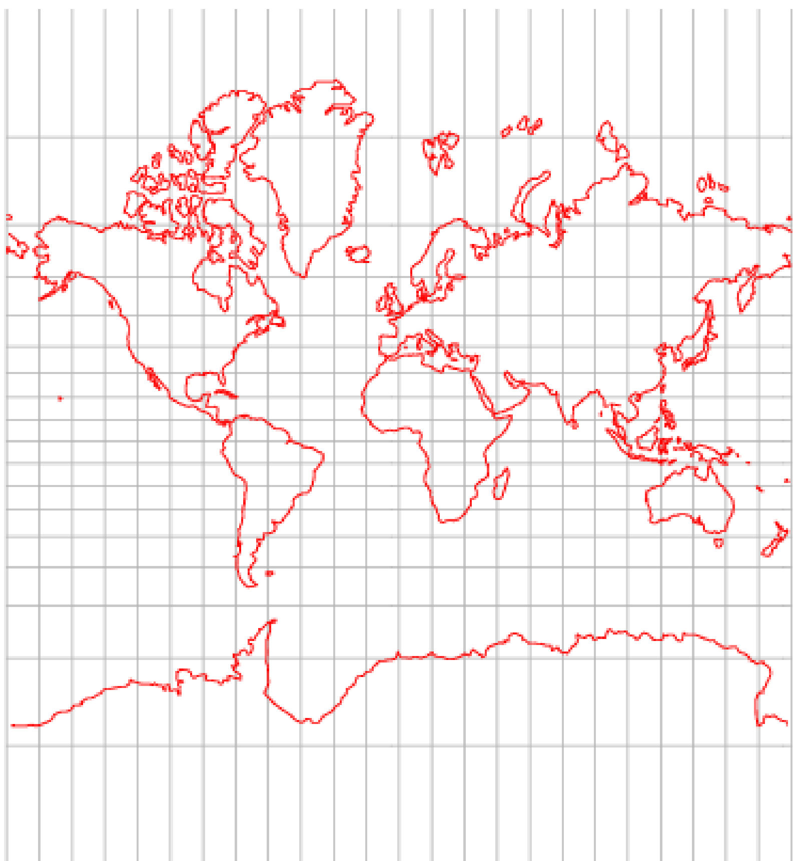

Figure 8: Mercator Projection.

12. (a) **Mercator Projection.** Gerhardt Mercator (1512–1594), a cartographer of the Flemish-Dutch school, employed this projection Fig. 12, for his famous map of the

world in 1569 (on which the term “Norumbega” was used to label a section of present New England.) He was aware that the projection preserves angles. Show that the Mercator Projection

$$\sigma(u,v) = (\operatorname{sech} u\,\cos v,\ \operatorname{sech} u\,\sin v,\ \tanh u)$$

gives a regular surface patch of the unit sphere. Show that the meridians and parallels on the sphere correspond under σ to perpendicular straight lines in the plane.

A *loxodrome* is a curve on the unit sphere that intersects each meridian at a fixed angle, say α. Show that in the Mercator surface patch σ, a unit speed loxodrome satrisfies

$$\dot u = \cos\alpha\,\cosh u, \qquad \dot v = \pm\sin\alpha\,\cosh u.$$

Deduce that loxodromes correspond under σ to straight lines of the uv-plane. [See exercises 4.19 and 4.20 of the text.]

(b) **Quadric Surfaces.** Find a rigid motion that reduces the following quadratic surface to one of the standard forms from Proposition 4.6. What kind of quadric surface is it?

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + x + 1 = 0.$$

Please hand in problems 13 on Monday, Feb. 25.

13. **First Fundamental Form.** Suppose $f, g : (\alpha,\beta) \to \mathbb{R}$ be smooth functions such that $f(t) > 0$ and $(f')^2 + (g')^2 = 1$. Let a surface of revolution $\mathcal{S}$ be given by $\sigma : (\alpha,\beta)\times\mathbb{R} \to \mathbb{R}^3$ where

$$\sigma(u,v) = (\,f(u)\,\cos v,\ f(u)\,\sin v,\ g(u)\,).$$

Find the first fundamental form.

Suppose that $\alpha < a \le b < \beta$ and consider the curve $\gamma(t) = \sigma(t, v_0)$ for $a \le t \le b$ and v_0 some fixed number. Show that if $\zeta : [a,b] \to \mathcal{S}$ is any other regular curve in the surface such that $\gamma(a) = \zeta(a)$ and $\gamma(b) = \zeta(b)$, then the lengths satisfy

$$\mathrm{L}(\gamma) \le \mathrm{L}(\zeta).$$

Show also that equality holds if and only if ζ is a reparametrization of γ.

Please hand in any three of problems 14, 15, 16a or 16b on Monday, Mar. 3.

14. **Isometries of Surfaces.** Suppose that the surface $\mathcal{C}$ is a cone with vertex angle $\alpha \in (0,\pi)$. It is given for $0 < u$ and $0 < v < 2\pi$ by

$$\sigma(u,v) = (\,u\,\sin\alpha\,\cos v,\ u\,\sin\alpha\,\sin v,\ u\cos\alpha\,).$$

Find a mapping from the cone to the Euclidean plane $f : \mathcal{C} \to \mathbb{E}^2$ that unwraps the cone. Verify that it is really an isometry. [See problem 5.5 on p.105 of the text.]

15. **First fundamental form under reparametrization.** [See ex. 100[5.4] of the text.]

Suppose $\mathcal{S}$ be a regular smooth surface, $W \subset \mathbb{R}^3$ an open set and $\sigma : U \to \mathcal{S} \cap W$ and $\tilde{\sigma} : \tilde{U} \to \mathcal{S} \cap W$ be two coordinate charts. Then the transition function $g : U \to \tilde{U}$ is given by $g(u,v) = \tilde{\sigma}^{-1} \circ \sigma(u,v) = \big(\tilde{u}, \tilde{v}\big)$ and so $\sigma = \tilde{\sigma} \circ g$. The Jacobian matrix is defined by

$$J = \begin{pmatrix} \dfrac{\partial \tilde{u}}{\partial u} & \dfrac{\partial \tilde{u}}{\partial v} \\ \dfrac{\partial \tilde{v}}{\partial u} & \dfrac{\partial \tilde{v}}{\partial v} \end{pmatrix}.$$

Show that the first fundamental forms in the σ and $\tilde{\sigma}$ charts are related by

$$\begin{pmatrix} E & F \\ & \\ F & G \end{pmatrix} = J^t \begin{pmatrix} \tilde{E} \circ g & \tilde{F} \circ g \\ & \\ \tilde{F} \circ g & \tilde{G} \circ g \end{pmatrix} J. \tag{1}$$

Suppose that $U, \tilde{U} \subset \mathbb{R}^2$ are two open sets and that $g : U \to \tilde{U}$ is a diffeomorphism given by $g(u,v) = (\tilde{u}, \tilde{v})$. Suppose that $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ and $\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix}$ are two positive definite matrix functions on U and $\tilde{U}$ that satisfy (1) on U. Let $\gamma : [a,b] \to U$ be a smooth curve given by $\gamma(t) = \big(u(t), v(t)\big)$ and $\tilde{\gamma}(t) = g \circ \gamma(t)$ the corresponding curve in $\tilde{U}$. Show that both curves have the same length $\mathrm{L}(\gamma) = \tilde{\mathrm{L}}(\tilde{\gamma})$. The lengths are defined by first fundamental forms

$$\mathrm{L}(\gamma) = \int_a^b \sqrt{E(u(t), v(t))\dot{u}^2(t) + 2F(u(t), v(t))\dot{u}(t)\dot{v}(t) + G(u(t), v(t))\dot{v}^2(t)}\, dt,$$
$$\tilde{\mathrm{L}}(\tilde{\gamma}) = \int_a^b \sqrt{\tilde{E}(\tilde{u}(t), \tilde{v}(t))\dot{\tilde{u}}^2(t) + 2\tilde{F}(\tilde{u}(t), \tilde{v}(t))\dot{\tilde{u}}(t)\dot{\tilde{v}}(t) + \tilde{G}(\tilde{u}(t), \tilde{v}(t))\dot{\tilde{v}}^2(t)}\, dt.$$

16. (a) **Area of a surface of revolution.** Let $f : (\alpha, \beta) \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be smooth functions such that $f > 0$ and $\dot{f}^2 + \dot{g}^2 = 1$. Let

$$\sigma(u,v) = \big(f(u)\, \cos v,\ f(u)\, \sin v,\ g(u)\big)$$

be a surface of revolution and for $\alpha < a \le b < \beta$ and $0 \le \theta_0 \le 2\pi$ be constants and $R = \{(u,v) : a \le u \le b,\ 0 \le v \le \theta_0\}$ be a region in the plane. Show that the area

$$\mathrm{A}(\sigma(R)) = \theta_0 \int_a^b f(u)\, du.$$

Compute the area of the circular cap of the unit sphere of angle $0 \le \omega \le \pi$,

$$C = \{(\sin u\, \cos v, \sin u\, \sin v, \cos u) : 0 \le u \le \omega,\ 0 \le v \le 2\pi\}.$$

Let $0 < b < a$. Compute the area of the torus T generated by rotating the circle $(x-a)^2 + z^2 = b^2$ in the xz-plane about the z-axis.

(b) **Second fundamental form of a graph.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and

$$\sigma(u,v) = \big(u,\ v,\ f(u,v)\big)$$

be a coordinate patch of a graph surface. Compute its second fundamental form.

Please hand in any two of problems 17, 18 or 19 on Monday, Mar. 10.

17. **Planes have zero second fundamental form.** Let $\mathcal{S}$ be a smooth, connected regular surface patch. Suppose that the second fundametal form vanishes everywhere on $\mathcal{S}$. Show that $\mathcal{S}$ is a (piece of a) plane. [Problem 126[6.2] of the text.]

18. **Normal and geodesic curvature.** Let $f, g : (\alpha, \beta) \to \mathbb{R}$ be smooth functions such that $f > 0$ and $\dot{f}^2 + \dot{g}^2 = 1$. Let

$$\sigma(u, v) = \big(f(u)\, \cos v,\ f(u)\, \sin v,\ g(u)\, \big)$$

be a surface of revolution. Find the normal and geodesic curvatures of the meridian and parallel curves on the surface. [Problem 129[6.9] of the text.]

19. **Principal curvatures of the other surface of revolution.** Let $f : (\alpha, \beta) \to \mathbb{R}$ be a smooth function where $\alpha \geq 0$. Another surface of revolution is given by

$$\sigma(u, v) = \big(u \cos v,\ u \sin v,\ f(u) \big).$$

Find the principal curvatures and principal directions of this σ.

Problem 20 and 21 are due on Monday, Mar. 10.

20. Let S be the surface

$$\sigma(u, v) = (u,\ v,\ uv\,)$$

and $P = (0, 0, 0)$. The vector $V(\theta) = (\cos\theta\,, \sin\theta,\ 0\,)$ is in the tangent plane $T_P S$. Compute the normal curvature $\kappa_n(\theta)$ of S in the $V(\theta)$ direction at P.

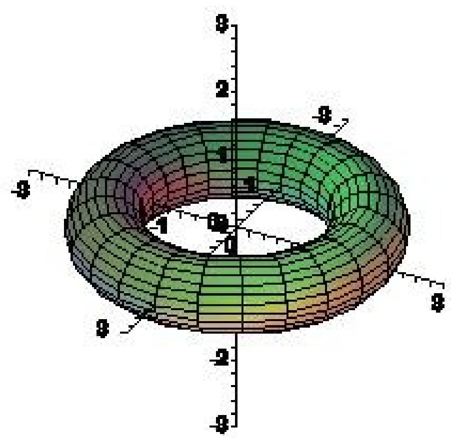

Figure 9: Torus.

21. **Lines of curvature.** (See problem 140[6.18] of the text.) A smooth curve $\mathcal{C}$ on a surface $\mathcal{S}$ is called a *line of curvature* if its tangent vectors are principal directions of $\mathcal{S}$ at all points of $\mathcal{C}$.

   (a) If the line of curvature is given in a coordinate patch by $\gamma : (\alpha, \beta) \to \mathcal{S}$, parameterized by arclength, then $\frac{d}{dt}\mathbf{N}(\gamma(t)) = -\kappa_1 \gamma'(t)$, where κ_1 is the principal curvature in the $\gamma'(t)$ direction.

   (b) If $\gamma : (\alpha, \beta) \to \mathcal{S}$ is any curve, parameterized by arclength, that satisfies $\frac{d}{dt}\mathbf{N}(\gamma(t)) = \lambda(t)\gamma'(t)$ at all t, then γ is a line of curvature and $\lambda = \kappa_1$ is the corresponding principal curvature.

(c) Let $0 < a < b$. Find the lines of curvature of the torus

$$(x^2 + y^2 + z^2 + b^2 - a^2)^2 = 4b^2(x^2 + y^2).$$

Problems 22 and 23 are due on Monday, Mar. 31.

22. Let $\mathcal{S}$ be a smooth closed, compact, connected surface. (*i.e.,* $\mathcal{S}$ is closed as a subset of $\mathbb{R}^3$, has no boundary and is bounded.) Show that $\mathcal{S}$ has an elliptic point. [Hint. Look at my solutions of the sample problems for the second exam.]

23. Let $0 < p < q < r$ and $\mathcal{S} = \left\{(x, y, z) \in \mathbb{R}^3 : \dfrac{x^2}{p^2} + \dfrac{y^2}{q^2} + \dfrac{z^2}{r^2} = 1\right\}$ be an ellipsoid. Show that $\mathcal{S}$ has exactly four umbillic points. (See problem 145[6.24] of the text.)

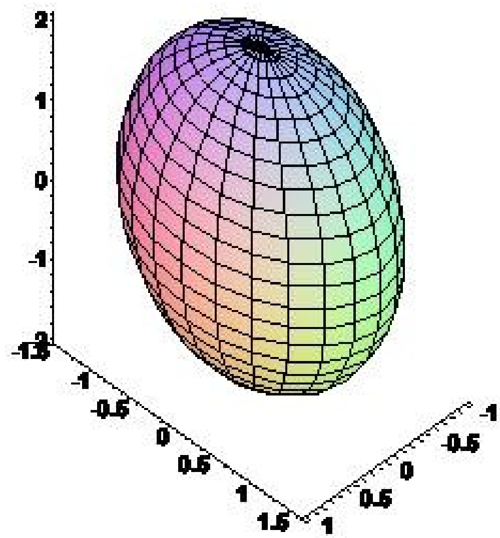

Figure 10: Ellipsoid for Problem 23.

Please hand in any three of problems 24, 25a, 25b or 26 on Monday, Apr. 7.

24. Let $U \in \mathbb{R}^2$ be an open set and $f : U \to \mathbb{R}$ a smooth function. Find the principal curvatures κ_1 and κ_2, the mean curvature H and the Gauss curvature K for the surface $z = f(x, y)$. Compute these for Sherk's first surface

$$z = \log\left(\frac{\cos y}{\cos x}\right).$$

[Problems 150[7.3] and 164[7.16] of the text.]

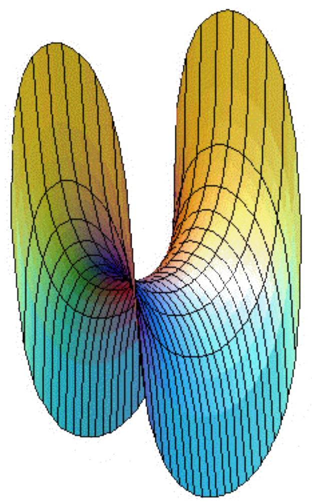

Figure 11: Scherk's First Surface.

25. (a) Let $\mathcal{S}$ be a smooth regular surface and $P \in \mathcal{S}$. Let $\mathbf{w} \in T_P\mathcal{S}$ be a unit vector and $\kappa_n(\mathbf{w})$ be the normal curvature of $\mathcal{S}$ at P in the $\mathbf{w}$ direction. Show that the mean curvature H at P has the alternative expressions

    i. $H = \frac{1}{2}\left(\kappa_n(\mathbf{w}_1) + \kappa_n(\mathbf{w}_2)\right)$, where $\mathbf{w}_1, \mathbf{w}_2 \in T_P\mathcal{S}$ are any two orthonormal vectors.

    ii. Fix a unit vector $\mathbf{w}_0 \in T_P\mathcal{S}$. Then

    $$H = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\mathbf{w}(\theta))\, d\theta,$$

    where $\mathbf{w}(\theta) \in T_P\mathcal{S}$ is a unit vector that makes an angle θ with $\mathbf{w}_0$.

    (b) Let $\gamma, \delta : (\alpha, \beta) \to \mathbf{R}^3$ be smooth such that γ is a regular space curve parameterized by arclength and $\|\delta\| = 1$ is a unit vector function. Assume that the *ruled surface* given by

    $$\sigma(u, v) = \gamma(u) + v\delta(u).$$

    is regular. Show that the Gauss curvature of σ vanishes $K \equiv 0$ if and only if the surface normal is constant along the rulings (the $v \mapsto \sigma(u, v)$ lines.) Such a surface is called a *developable surface.*

26. Find $\int_{\mathcal{S}} K\,dA$ where $\mathcal{S}$ is *Enneper's Surface,* given for all $(u,v) \in \mathbb{R}^2$ by

$$\sigma(u,v) = \left(\ u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + u^2 v,\ u^2 - v^2\ \right).$$

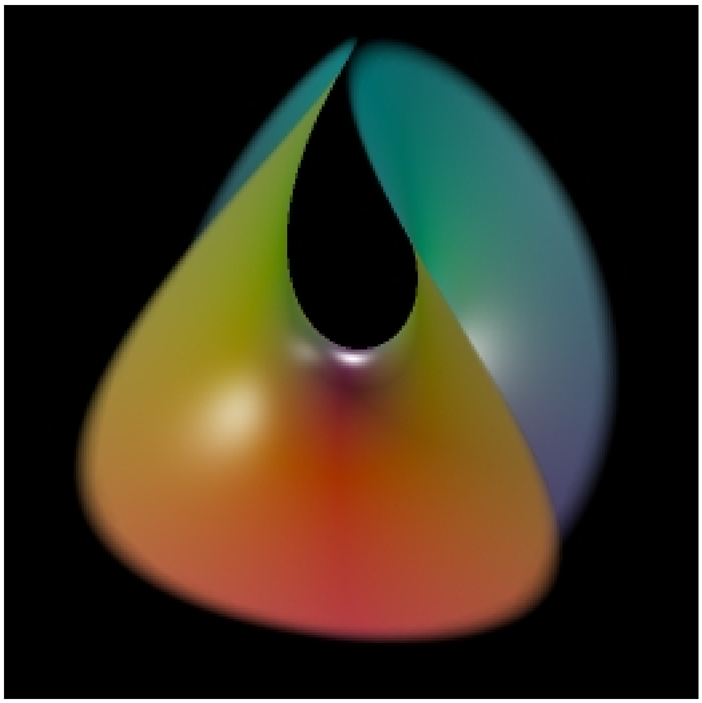

Figure 12: Enneper's Surface near the origin. (See p. 214 for a wider view).

Please hand in any three of problems 27, 28, 29 or 30 on Monday, Apr. 14.

27. Find the Gauss curvature K using Gauss's formula for the metric on the upper halfplane $\{(u,v) \in \mathbb{R}^2 : v > 0\}$ given by

$$ds^2 = \frac{du^2 + dv^2}{v^2}.$$

28. Two isometric surfaces have the same Gaussian curvature according to Gauss's Theorem. This exercise shows that having identical Gauss curvatures does not mean that the surfaces are isometric. (Exercise 240[10.7] from the text.)

    (a) Let $\sigma, \tilde{\sigma} : \mathcal{U} \to \mathbb{R}^3$, where $\mathcal{U} = \{(u,v) \in \mathbb{R}^2 :\ u > 0 \text{ and } -\pi < v < \pi\}$ be given by

    $$\sigma(u,v) = (\ u\cos v,\ u\sin v,\ \ln u\ )$$
    $$\sigma(u,v) = (\ u\cos v,\ u\sin v,\ v\ )$$

    Show that σ and $\tilde{\sigma}$ have identical Gaussian curvatures at corresponding points.

    (b) Show that the map $\sigma(u,v) \mapsto \tilde{\sigma}(u,v)$ is not an isometry.

    (c) Show that there is no isometry from σ to $\tilde{\sigma}$ (even if one restricts to smaller $\mathcal{U}$ and maps to some other part of $\tilde{\sigma}$).

29. Let $\sigma(u_1, u_2)$ be a coordinate patch of the surface $\mathcal{S}$ and $g_{ij} = \sigma_i \bullet \sigma_j$ be the first fundamental form of the surface. Let $g^{ij} = (g_{ij})^{-1}$ be the inverse matrix.

    (a) Find the derivatives of the inverse matrix

    $$\frac{\partial}{\partial u_k} g^{ij}$$

    in terms of g_{ij} and its derivatives $\frac{\partial}{\partial u_k} g_{ij}$.

    (b) Find the equations that result by equating coefficients of $\{\sigma_1, \sigma_2, \mathbf{N}\}$, in the equations

    $$\mathbf{N}_{ij} = \mathbf{N}_{ji}.$$

    (c) Determine whether the resulting equations yield any new relations. Or, could they have already been deduced from definitions and the Codazzi equations?

30. Given two quadratic forms, determine whether they can be the first and second fundamental forms of a surface. If so, find the surface. (Exercise 244[10.9-10] from the text.)

    (a) $\cos^2 v\, du^2 + dv^2$ and $-\cos^2 v\, du^2 - dv^2$;

    (b) $du^2 + \cos^2 u\, dv^2$ and $\cos^2 u\, du^2 + dv^2$.

Please hand in any three problems from 31, 32, 33, 34 or 35 on Monday, Apr. 21.
The **Final Exam** is Tuesday, April 29 at 8:00 – 10:00 am.
No homework will be accepted after the final.

31. **Geodesics are intrinsic to the surface.** Let $\sigma : \mathcal{U} \to \mathcal{S} \subset \mathbb{R}^3$ be a patch for the surface, and $\gamma : (\alpha, \beta) \to \mathcal{S}$ be a smooth unit speed curve given by

    $$\gamma(t) = \sigma\big(\ u(t),\ v(t)\ \big).$$

    The geodesic curvature is the tangential part of the acceleration of γ viewed as a space curve, so

    $$\kappa_g := \ddot{\gamma} \bullet (\mathbf{N} \times \dot{\gamma}).$$

    The curve is a geodesic if $\kappa_g \equiv 0$. Show that the geodesic curvature may be expressed in terms of just the first fundamental form of the surface by the formula

    $$\kappa_g = (\dot{u}\,\ddot{v} - \ddot{u}\,\dot{v})\sqrt{EG - F^2} + A\dot{u}^3 + B\dot{u}^2\dot{v} + C\dot{u}\dot{v}^2 + D\dot{v}^3,$$

    where A, B, C, D are functions of E, F, G and their derivatives. (Exercise 181[8.11] from the text.)

32. Let $\mathcal{S}$ be a smooth surface with positive Gaussian curvature, $K > 0$, and without umbillic points. Show that there is no point of $\mathcal{S}$ where H is a maximum and K is a minimum.

33. Let $\mathcal{S}$ be the surface of the two-holed torus. Find a triangulation of $\mathcal{S}$ and compute the Euler characteristic directly from the triangulation.

    $$\chi = E - V + F.$$

34. Show that if a a cylinder is given a metric of negative curvature, then it has at most one closed geodesic. Show that the statement is false if "negative" is replaced by "non-positive."

35. Show that the bipartite graph $B(3, 3)$ cannot be embedded in the plane. (You will have to correct the argument given in the solution to exercise 269[11.8] from the text. See also problem 11.7, that the complete graph on five vertices can't be embedded either.)

Figure 13: Two-holed Torus

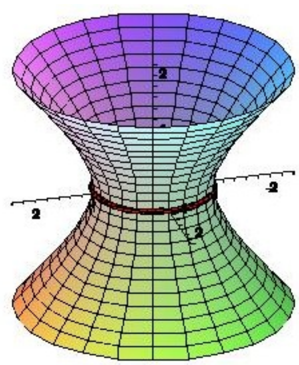

Figure 14: Negative curvature cylinder with closed geodesic

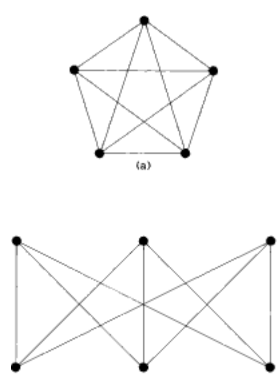

Figure 15: (a.) Complete graph on five vertices $K(5)$ (b.) Bipartite graph $B(3,3)$.