5010 solutions, Assignment 14. Chapter 8: 1–5, 13, 16, 19, 27.

1. We need $f(x,y) \ge 0$ and $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = 1$. The integral is equal to

$$\frac{1}{4} + \frac{1}{2}(a+b) + 1,$$

so for this to be 1 we need $a+b = -1/2$. Under what conditions is $f(x,y) \ge 0$ for all x, y? Now f has an interior minimum at (x,y) only if the two partials are 0 there, that is, $y+a=0$ and $x+b=0$. This would require $x=-b \in (0,1)$ and $y = -a \in (0,1)$, so $f(x,y) = ab - ab - ab + 1 = 1 - ab > 0$.

So we have to examine f only on the boundary of the square. Now $f(x,0) = ax+1 \ge 0$ if $a \ge -1$ and $f(0,y) = by+1 \ge 0$ if $b \ge -1$. Also $f(x,1) = x + ax + b + 1 \ge 0$ since it is nonnegative when $x=0$ (if $b \ge -1$) and $x=1$ (since $a+b=-1/2$), and $f(1,y) = y + a + by + 1 \ge 0$ since it is nonnegative when $y=0$ (if $a \ge -1$) and $y=1$ (since $a+b=-1/2$).

We conclude that the conditions on a and b are $a \ge -1$, $b \ge -1$, and $a+b=-1/2$. (Book's answer replaces these inequalities by $a \ge -2$ and $b \ge -2$. Take $a=-2$ and $b=3/2$, for example. Then $f(1,0) = a+1 = -2+1 = -1$. So book's answer is wrong.)

Can we ever have independence? We can evaluate the marginal densities, namely $f_X(x) = (a+1/2)x + (1/2)b + 1$ and $f_Y(y) = (b+1/2)y + (1/2)a + 1$, and check that their product cannot be $f(x,y)$. Indeed, for this to be true, we would need $(a+1/2)(b+1/2) = 1$ or $1 = ab + (1/2)(a+b) + 1/4 = ab$, and this is impossible.

2. $E[XY] = \int_0^1 \int_0^1 xy(xy+ax+by+1)\,dx\,dy = (1/3)^2 + (1/3)(1/2)(a+b) + (1/2)^2 = (4+6(a+b)+9)/36 = (4+6(-1/2)+9)/36 = 10/36 = 5/18$. $E[X] = \int_0^1 x[(a+1/2)x + (1/2)b+1]\,dx = (1/3)(a+1/2) + (1/2)((1/2)b+1) = (4a+3b+8)/12 = (-b+4(-1/2)+8)/12 = (6-b)/12$ and $E[Y] = \int_0^1 y[(b+1/2)y + (1/2)a+1]\,dy = (1/3)(b+1/2) + (1/2)((1/2)a+1) = (4b+3a+8)/12 = (-a+4(-1/2)+8)/12 = (6-a)/12$. The final result is $\mathrm{Cov}(X,Y) = 5/18 - (6-a)(6-b)/(12)^2 = (40-(36-6(a+b)+ab))/144 = (40-36+6(-1/2)-ab))/144 = (1-ab)/144$.

3. (a) Since e^{-x} $(x>0)$ is a density, $c=1$.

(b) $P(X+Y>1) = 1 - P(X+Y \le 1) = 1 - \int_0^1 \int_0^{1-y} e^{-x-y}\,dx\,dy = 1 - \int_0^1 (1-e^{-(1-y)})e^{-y} dy = 1 - \int_0^1 (e^{-y} - e^{-1})\,dy = 1 - (1 - e^{-1} - e^{-1}) = 2e^{-1}$.

(c) $P(X<Y) = 1/2$ by symmetry.

4. $F_Z(z) = P(X+Y \le z) = \int_0^z \int_0^{z-x} g(x+y)\,dy\,dx = \int_0^z \int_x^z g(y)\,dy\,dx = \int_0^z \int_0^y g(y)\,dx\,dy = \int_0^z y g(y)\,dy$, hence $f_Z(z) = zg(z)$, $z>0$.

5. (a) $\int_{-\infty}^\infty \int_{-\infty}^\infty c(1+x^2+y^2)^{-3/2}\,dx\,dy = \int_0^{2\pi} \int_0^\infty c(1+r^2)^{-3/2}\,r\,dr\,d\theta = 2\pi c \int_0^\infty (1+r^2)^{-3/2}\,r\,dr = 2\pi c$, so $c = (2\pi)^{-1}$.

(b) $f_X(x) = (2\pi)^{-1} \int_{-\infty}^\infty (1+x^2+y^2)^{-3/2}\,dy$. Here we use the formula shown in the back of the book with $a = 1+x^2$ to get $f_X(x) = 2(2\pi)^{-1}(1+x^2)^{-1} = 1/[\pi(1+x^2)]$.

13. The marginal of Y is $f_Y(y) = \int_0^y cye^{-y(x+1)}\,dx = cye^{-y}\int_0^y e^{-yx}\,dx = cye^{-y}(1-e^{-y^2})/y = ce^{-y}(1-e^{-y^2})$. Now divide this into the joint density to get $f_{X|Y}(x|y) = ye^{-yx}/(1-e^{-y^2})$, $0 < x < y < \infty$. Note that we didn't need to evaluate c.

16. (a) $P(U = X) = P(X < Y) = \int_0^\infty \int_x^\infty \lambda e^{-\lambda x}\mu e^{-\mu y}\,dy\,dx = \int_0^\infty \lambda e^{-\lambda x}e^{-\mu x}\,dy\,dx = \lambda/(\lambda+\mu)$.

(b) U and $V-U$ are independent by the lack-of-memory property. Think of X and Y as the times at which independent alarms ring. The first to ring is at time U, the second to ring is at time V. By the lack of memory property, the conditional distribution of the time between rings, given the time of the first ring, does not depend on the time of the first ring.

19. $P(T \geq j+1) = P(X_1+\cdots+X_j < 1) = \int_0^1\int_0^{1-x_1}\cdots\int_0^{1-x_1-\cdots-x_{j-1}} dx_j\,dx_{j-1}\cdots dx_1 = \int_0^1\int_0^{1-x_1}\cdots\int_0^{1-x_1-\cdots-x_{j-2}}(1-x_1-\cdots-x_{j-1})\,dx_{j-1}\cdots dx_1 = \int_0^1\int_0^{1-x_1}\cdots\int_0^{1-x_1-\cdots-x_{j-3}}[(1-x_1-\cdots-x_{j-1})^2/2]\,dx_{j-1}\cdots dx_1 = \cdots = 1/j!$. Hence $E[T] = \sum_{j\geq 0} 1/j! = e$.

27. $X - Y$ has the same distribution as $\sum_{i=1}^n (X_i - Y_i)$, where $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are independent Poisson(1) random variables. $X_1 - Y_1$ has mean 0 and variance 2, so

$$\frac{\sum_{i=1}^n (X_i - Y_i)}{\sqrt{2n}} \to_d N(0,1)$$

by the central limit theorem. This is equivalent to the statement of the problem.