5010 solutions, Assignment 3. Chapter 2: 11, 12, 15, 19, 20, 23, 27, 34, 35.

11. Set up the problem as in the solutions, page 482. We need $P(F_r \mid G) = P(F_r)P(G \mid F_r)/P(G) = (1/3)(1-(1/2)^r)/[(1/3)(1-(1/2)^1)+(1/3)(1-(1/2)^2)+(1/3)(1-(1/2)^3)]$, which is $(1/2)/(1/2+3/4+7/8)$, $(3/4)/(1/2+3/4+7/8)$, $(7/8)/(1/2+3/4+7/8)$, or 4/17, 6/16, or 7/17. Next, $P(N \mid F_1 \cap G) = 1$, obviously. $P(N \mid F_2 \cap G)$ can be obtained as follows: F_2 says family must be GG, GB, BG, or BB. Event G rules out BB. If a family is chosen at random and the a girl in the family is chosen at random, then N occurs with probability 5/6. $P(N \mid F_3 \cap G)$ can be obtained as follows: F_3 says family must be GGG, GGB, GBG, GBB, BGG, BGB, BBG, or BBB. Event G rules out BBB. If a family is chosen at random and the a girl in the family is chosen at random, then N occurs with probability $(1/7)(1/3+1/2+1/2+1+1/2+1+1) = (1/7)(29/6) = 29/42$.

12. (a) Probability that I can drive to Beaton is $1-p^2$. Probability that I can drive from Beaton to City is also $1-p^2$. Result is $(1-p^2)^2$.

(b) This is the probability that I can travel by train, or the railway is blocked but I can travel by car, hence $1-p+p(1-p^2)^2$.

(c) By the definition of conditional probability, this is the probability that the railway is blocked but I can travel by car, divided by the probability that I can travel to the City. $p(1-p^2)^2/[1-p+p(1-p^2)^2]$.

15. The solution on page 482 explains how to do it.

19. (a) Think of the games as occurring in pairs. Before a pair of games, A and B are tied. They remain tied with win-loss or loss-win. But with win-win for A, A wins, and with loss-loss for A, B wins. So there are 3 outcomes of a pair of games, win-win (probability p^2), loss-loss (probability q^2), and one of each (probability $2pq$). So the probability that A wins is the probability that win-win occurs before loss-loss, namely,

$$\frac{p^2}{p^2+q^2} = \frac{p^2}{1-2pq}.$$

(b) Let E_n be the event that A wins the sequence at the nth game. Then

$$P(E_{2n}) = (pq)^{n-1}p^2, \qquad P(E_{2n+1}) = q(pq)^{n-1}p^2,$$

and the sum over all $n \geq 1$ is $(1+q)p^2/(1-pq)$.

(c) Let P_1 and P_2 denote the two probabilities of A winning the sequence. Assume A is the weaker player ($p < \frac{1}{2}$). Method 2 will be better if $P_2/P_1 > 1$, or if $P_2/P_1 = (1+q)(1-2pq)/(1-pq) > 1$, or if $(1+q)(1-2pq) > (1-pq)$, or if $1+q-2pq-2pq^2 > 1-pq$, or if $q-pq-2pq^2 > 0$, or if $1-p-2pq > 0$, or if $q-2pq > 0$, or if $1-2p > 0$, which is true by assumption.

20. Let H be the event that the first coin is heads. Let A_2 be the event that your score is 2. We want $P(H^c \mid A_2)$.

$$\begin{aligned} P(H^c \mid A_2) &= \frac{P(H^c)P(A_2 \mid H^c)}{P(H)P(A_2 \mid H) + P(H^c)P(A_2 \mid H^c)} \\ &= \frac{(1/2)(1/6)}{(1/2)(1/6)+(1/2)(5/32)} = 16/31. \end{aligned}$$

23. The probability of a hit on the first throw is $1/4$, on the second throw is $(1/4)(2/3)$, on the third throw is $(1/4)(2/3)^2$, and so on, so the probability of a hit on the nth throw is $(1/4)(2/3)^{n-1}$. Probability of eventual hit is $\sum_{n=1}^{\infty}(1/4)(2/3)^{n-1} = (1/4)/(1-2/3) = 3/4$, or actually less than this since Irena will give up eventually, so the probability of never hitting greater than $1/4$.

27. (a) Requires $n-1$ failures followed by a success. $(5/6)^{n-1}(1/6)$.

(b) $\sum_{n=1}^{\infty}(5/6)^{2n-1}(1/6) = (1/5)\sum_{n=1}^{\infty}(25/36)^n = (1/5)(25/36)/(1-25/36) = 5/11$.

(c) Probability of no 5 before first 6 is the probability of the first 6 before the first 5, which is $(1/6)/(1/6+1/6) = 1/2$, so the complementary probability is also $1/2$.

Next the probability of 1 before 2–6 and 2 before 3–6 and 3 before 4–6 and 4 before 5–6 and 5 before 6 is

$$\frac{1}{6}\,\frac{1/6}{1/6+4/6}\,\frac{1/6}{1/6+3/6}\,\frac{1/6}{1/6+2/6}\,\frac{1/6}{1/6+1/6} = \frac{1}{6!}$$

and there are $5!$ ways of permuting the numbers 1–5, so the result is $5!/6! = 1/6$.

34. Answers on page 483 are clear.

35. (a)

$$\begin{aligned} P(M \mid C) &= \frac{P(M)P(C \mid M)}{P(M)P(C \mid M) + P(F)P(C \mid F)} \\ &= \frac{(200/1300)(110/200)}{(200/1300)(110/200) + (1100/1300)(120/1100)} = \frac{11}{23}. \end{aligned}$$

(b) $1 - 11/23 = 12/23$.