5010 solutions, Assignment 8. Chapter 5: 1, 3, 6, 9, 10, 12, 18, 21, 24.

1. $f(x,y) = \binom{2}{x,y,2-x-y}(1/6)^x(1/6)^y(4/6)^{2-x-y} = \binom{2}{x,y,2-x-y}4^{2-x-y}/36$ for $x \geq 0$, $y \geq 0$, and $x+y \leq 2$. We find that $f(0,0) = 16/36$, $f(0,1) = f(1,0) = 8/36$, $f(1,1) = 2/36$, and $f(0,2) = f(2,0) = 1/36$. The marginals are both binomial$(2,1/6)$ so $E[X] = E[Y] = 1/3$ and $\mathrm{Var}(X) = \mathrm{Var}(Y) = 2(1/6)(5/6) = 5/18$. Next $E[XY] = 0(34/36)+1(2/36) = 1/18$, so $\mathrm{Cov}(X,Y) = 1/18-(1/3)^2 = -1/18$ and $\rho(X,Y) = (-1/18)/(5/18) = -1/5$.

3. Let X and Y be the results of two dice throws. The $X+Y$ and $X-Y$ are uncorrelated but not independent. Indeed, $\mathrm{Cov}(X+Y, X-Y) = E[(X+Y)(X-Y)] = E[X^2]-E[XY]+E[YX]-E[Y^2] = E[X^2]-E[Y^2] = 0$ and, if $X+Y = 2$, then $X = Y = 1$, so $X-Y = 0$. The example in the book's answers appears wrong because Y is constant, and a constant is independent of any random variable.

6. The values of Y are $0,1,0,-1,0,1,0,-1,\ldots$, each with probability $1/(4n)$. The values of Z are $1,0,-1,0,1,0,-1,0,\ldots$, each with probability $1/(4n)$. The joint probability mass function of Y and Z is $f(0,1) = 1/4$, $f(1,0) = 1/4$, $f(0,-1) = 1/4$, and $f(-1,0) = 1/4$. Therefore $Y+Z$ is ± 1 with probability $1/2$ each. Orthogonality means $E[YZ] = 0$ and this expectation is 0 because YZ is the 0 random variable.

9. (a) $P(U=n) = P(X_1=1)P(U=n \mid X_1=1)+P(X_1=-1)P(U=n \mid X_1=-1) = pp^{n-1}q+qq^{n-1}p$, which is p times a geometric(q) distribution plus q times a geometric(p) distribution. So $E[U] = p(1/q)+q(1/p)$. $P(V=n) = P(X_1=1)P(V=n \mid X_1=1)+P(X_1=-1)P(V=n \mid X_1=-1) = pq^{n-1}p+qp^{n-1}q$, which is p times a geometric(p) distribution plus q times a geometric(q) distribution. So $E[V] = p(1/p)+q(1/q) = 2$.

(b) $P((U,V)=(m,n)) = P(X_1=1)P((U,V)=(m,n) \mid X_1=1)+P(X_1=-1)P((U,V)=(m,n) \mid X_1=-1) = pp^{m-1}q^np+qq^{m-1}p^nq = p^{m+1}q^n+q^{m+1}p^n$ for $m,n \geq 1$. Next,

$$\begin{aligned}
E[UV] &= p\sum_{m,n\geq 1} mn\cdot p^{m-1}q\cdot q^{n-1}p + q\sum_{m,n\geq 1} mn\cdot q^{m-1}p\cdot p^{n-1}q \\
&= pE[\text{geometric}(p)]E[\text{geometric}(q)] + qE[\text{geometric}(q)]E[\text{geometric}(p)] \\
&= \frac{p}{pq}+\frac{q}{qp} = \frac{1}{p}+\frac{1}{q}.
\end{aligned}$$

Hence

$$\mathrm{Cov}(U,V) = \frac{1}{p}+\frac{1}{q}-2\left(\frac{p}{q}+\frac{q}{p}\right) = \frac{1-2(p^2+q^2)}{pq} = \frac{4pq-1}{pq}.$$

The correlation requires finding the variances of U and V, which can be done by the same methods as above.

10. $P((X,Y)=(i,j)) = 1/[n(n-1)]$ for all $i \neq j$. So the joint distribution is uniform over the pairs (i,j) with $i,j = 1,\ldots,n$ and $i \neq j$. It follows that (or

it is obvious, à priori) that X and Y are uniform over $1, 2, \ldots, n$ and $E[U] = E[V] = (1 + 2 + \cdots + n)/n = (n+1)/2$. So

$$\begin{aligned} E[XY] &= \frac{1}{n(n-1)} \sum_{i \neq j} ij = \frac{1}{n(n-1)} \left(\sum_{i,j} ij - \sum_i i^2 \right) \\ &= \frac{1}{n(n-1)} \left(\left(\frac{n(n+1)}{2} \right)^2 - \frac{n(n+1)(2n+1)}{6} \right) = \frac{(n+1)(3n+2)}{12} \end{aligned}$$

hence

$$\operatorname{Cov}(X, Y) = \frac{(n+1)(3n+2)}{12} - \frac{(n+1)^2}{4} = -\frac{n+1}{12}.$$

Next, $E[U^2] = n(n+1)(2n+1)/(6n) = (n+1)(2n+1)/6$, so $\operatorname{Var}(U) = (n+1)(2n+1)/6 - (n+1)^2/4 = (n-1)(n+1)/12$. Hence $\rho(X, Y) = -1/(n-1)$. This tends to 0 as $n \to \infty$.

12. (a) $f(i, j) = 1/36$ if $i < j$, $f(i, i) = (6 - i + 1)/36$, $f(i, j) = 0$ if $i > j$.
(b) $f(i, j) = 2/36$ if $i < j$, $f(i, i) = 1/36$, $f(i, j) = 0$ if $i > j$. (done in class)
(c) $f(i, j, k) = 1/36$ if $k = \max(i, j)$, $f(i, j, k) = 0$ otherwise.

For $\operatorname{Cov}(U, V)$, we need $E[UV]$. There is a trick to get this one: Just note that $UV = XY$ and so $E[UV] = E[XY] = E[X]E[Y] = (7/2)^2$. Next $E[U] = (11 \cdot 1 + 9 \cdot 2 + 7 \cdot 3 + 5 \cdot 4 + 3 \cdot 5 + 1 \cdot 6)/36 = 91/36$ and $E[V] = (1 \cdot 1 + 3 \cdot 2 + 5 \cdot 3 + 7 \cdot 4 + 9 \cdot 5 + 11 \cdot 6)/36 = 161/36$. So $\operatorname{Cov}(U, V) = 49/4 - (91/36)(161/36) = (36 \cdot 9 \cdot 49 - 91 \cdot 161)/6^4 = 1225/6^4 = 5^2 7^2/6^4$. Finally,

$$E[XYV] = \frac{1}{36} \sum_{i=1}^{6} \sum_{j=1}^{6} ij \max(i, j)$$

This would be a bit tedious to compute by hand (and unreliable), so

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FOR i=1 to 6
    FOR j=1 to 6
        LET sum=sum+i*j*max(i,j)
    NEXT j
NEXT i
PRINT sum
END
 2275
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The answer is $2275/36 = 91(5/6)^2 = 63.19\overline{4}$.

18. (a)

$$\begin{aligned} E[Z] = E[\min(c, X)] &= \sum_{n=1}^{\infty} P(\min(c, X) \geq n) = \sum_{n=1}^{c} P(X \geq n) \\ &= \sum_{n=1}^{c} (1 - p_1)^{n-1} = \frac{1 - (1 - p_1)^c}{p_1}. \end{aligned}$$

Book's answer is wrong.

(b) $P(\min(X,Y) \geq n) = P(X \geq n, Y \geq n) = P(X \geq n)P(Y \geq n) = (1-p_1)^{n-1}(1-p_2)^{n-1} = [(1-p_1)(1-p_2)]^{n-1}$, and this implies that $\min(X,Y)$ has a geometric distribution with parameter $1-(1-p_1)(1-p_2) = p_1+p_2-p_1p_2$, hence its mean is the reciprocal of this, $1/(p_1+p_2-p_1p_2)$. Book's answer is wrong.

21. (a) $f_{X+Y}(z) = \sum_x f_X(x)f_Y(z-x) = (n+1)^{-2}(z+1)$ for $z = 0, 1, \ldots, n$ and $f_{X+Y}(z) = (n+1)^{-2}(n-(z-n)) = (n+1)^{-1}(2n-z+1)$ for $z = n, n+1, \ldots, 2n$.

(b) $f_{X-Y}(z) = \sum_x f_X(z+x)f_Y(x) = (n+1)^{-2}(\min(n, n-z) - \max(0, -z) + 1)$ for $z = -n, -n+1, \ldots, n-1, n$. For $z = 0, 1, \ldots, n$, this gives $f_{X-Y}(z) = (n+1)^{-2}(n-z+1)$. For $z = -n, -n+1, \ldots, 0$, this gives $f_{X-Y}(z) = (n+1)^{-2}(n+z+1)$.

24.

$$\begin{aligned} P(X+Y=n) &= \sum_m P(X=m,\, Y=n-m) = \sum_{m=0}^{n} (e^{-\lambda}\lambda^m/m!)(e^{-\mu}\mu^{n-m}/(n-m)!) \\ &= e^{-(\lambda+\mu)}\left[\sum_{m=0}^{n}\binom{n}{m}\lambda^m\mu^{n-m}\right]/n! = e^{-(\lambda+\mu)}(\lambda+\mu)^n/n!. \end{aligned}$$

$$\begin{aligned} P(X=k \mid Z=n) &= \frac{P(X=k,\, Z=n)}{P(Z=n)} = \frac{P(X=k,\, Y=n-k)}{P(Z=n)} \\ &= \frac{(e^{-\lambda}\lambda^k/k!)(e^{-\mu}\mu^{n-k}/(n-k)!)}{e^{-(\lambda+\mu)}(\lambda+\mu)^n/n!} = \binom{n}{k}p^k(1-p)^{n-k}, \end{aligned}$$

where $p = \lambda/(\lambda+\mu)$.