Math 5010 § 1. **Second Midterm Exam** Name: SomeSolutions
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Solutions to the Extra Sample Problems.

1. *Suppose that there are a amber and b beryl balls in an urn. Suppose that $k \le a+b$ are randomly withdrawn without replacement. Let X be the number of amber balls withdrawn. Using the method of indicators or otherwise, find* $\mathbf{E}(X)$ *and* $\mathbf{Var}(X)$.

   X is distributed according to the hypergeometric distribution. If $p = a/(a+b)$ is the fraction of amber balls initially, then the answers can be compared to those for the bimonial rv with $Y \sim \text{bin}(k,p)$. One way to do the problem is to take the pmf

$$f_X(x) = \frac{\binom{a}{x}\binom{b}{k-x}}{\binom{a+b}{k}}, \qquad \text{for } 0 \le x \le k.$$

   and use it to compute $\mathbf{E}(X)$ and $\mathbf{Var}(X) = \mathbf{E}(X^2) - \mathbf{E}(X)^2$ in the usual way. However, the method of indicators is far easier. Let $I_j = 1$ whenever the jth ball drawn is amber, and $I_j = 0$ otherwise. Then

$$X = \sum_{j=1}^k I_j \qquad \text{and} \qquad \mathbf{E}(X) = \sum_{j=1}^k \mathbf{E}(I_j).$$

   Without knowing anything else about the draws, the probabilities are the same for each draw

$$\mathbf{E}(I_j) = \mathbf{E}(I_1) = \frac{a}{a+b} = p$$

   for all j. Hence

$$\mathbf{E}(X) = \sum_{j=1}^k \mathbf{E}(I_j) = \sum_{j=1}^k \frac{a}{a+b} = \frac{a\,k}{a+p} = kp = \mathbf{E}(Y).$$

   To compute $\mathbf{E}(X^2)$ we shall need for $i \ne j$ the probability $\mathbf{E}(I_i I_j)$ that both the ith and the jth draws are amber. Again, since the probability for any two pairs is as likely as any other two,

$$\mathbf{E}(I_i I_j) = \mathbf{E}(I_1 I_2) = \frac{a(a-1)}{(a+b)(a+b-1)}.$$

   for all $i \ne j$. Hence using $X^2 = \left(\sum_{i=1}^k I_i\right)\left(\sum_{j=1}^k I_j\right) = \sum_{i=1}^k \sum_{j=1}^k I_i I_j$,

$$\begin{aligned}\mathbf{E}(X^2) &= \mathbf{E}\left(\sum_{i,j=1}^k I_i I_j\right) = \sum_{i,j=1}^k \mathbf{E}\left(I_i I_j\right) = \sum_i^k \mathbf{E}\left(I_i^2\right) + \sum_{i\ne j} \mathbf{E}\left(I_i I_j\right) \\ &= \frac{a\,k}{a+b} + \frac{a(a-1)\,k(k-1)}{(a+b)(a+b-1)} = \frac{ak}{a+b}\left(\frac{a+b-1+(a-1)(k-1)}{a+b-1}\right)\end{aligned}$$

   since $I_i^2 = I_i$. The variance is thus

$$\begin{aligned}\mathbf{Var}(X) = \mathbf{E}(X^2) - \mathbf{E}(X)^2 &= \frac{ak}{a+b}\left(\frac{b+ak-k}{a+b-1} - \frac{ak}{a+b}\right) \\ &= \frac{abk(a+b-k)}{(a+b)^2(a+b-1)} = kpq\left(\frac{a+b-k}{a+b-1}\right).\end{aligned}$$

   The variance of the hypergeometric is the same as the binomial $\mathbf{Var}(Y) = kpq$ except for the correction factor $(a+b-k)/(a+b-1)$ which is negligible when k is small compared to $a+b$.

2. *Let X and Y be independent random variables, such that X is uniform on $\{1,2,3,\ldots,m\}$ and Y is uniform on $\{1,2,3,\ldots,n\}$ where $1 \le m \le n$. Let $Z = X+Y$. Find the pmf $f_Z(z)$, $\mathbf{E}(Z)$ and $\mathbf{Var}(Z)$.*

   Since X and Y are uniform, their pmf's are

$$f_X(x) = \begin{cases} \frac{1}{m}, & \text{if } 1 \le x \le m; \\ 0, & \text{otherwise.} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{n}, & \text{if } 1 \le y \le n; \\ 0, & \text{otherwise.} \end{cases}$$

   Since X and Y are indepedent, the joint pmf $f(x,y) = f_X(x)f_Y(y)$. Using $m \le n$ we see that the pmf for $Z = X+Y$ is given for $z \in \{2,3,\ldots,m+n\}$ by

$$\begin{aligned} f_Z(z) &= \mathbf{P}(X+Y=z) = \sum_{x+y=z} f_X(x)f_Y(y) \\ &= \begin{cases} \sum_{j=1}^{z-1} f_X(j)f_Y(z-j), & \text{if } z \le m; \\ \sum_{j=1}^{m} f_X(j)f_Y(z-j), & \text{if } m < z \le n; \\ \sum_{j=z-n}^{m} f_X(j)f_Y(z-j), & \text{if } n < z; \end{cases} = \begin{cases} \frac{z-1}{mn} & \text{if } z \le m; \\ \frac{1}{n}, & \text{if } m+1 \le z \le n+1; \\ \frac{m+n-z+1}{mn}, & \text{if } n+1 < z; \end{cases} \end{aligned}$$

   Notice that the distribution is symmetric about $(m+n+2)/2$ so that $\mathbf{E}(Z) = (m+n+2)/2$. To see this, observe that $f_Z(z) = f_Z(m+n+2-z)$. Thus

$$\begin{aligned} 2\,\mathbf{E}(Z) &= 2\sum_{k=2}^{m+n} kf_Z(k) = \sum_{k=2}^{m+n} \left(kf_Z(k) + kf_Z(m+n+2-k)\right) \\ &= \sum_{k=2}^{m+n} \Big(kf_Z(k) + (m+n-k+2)f_Z(k)\Big) = m+n+2. \end{aligned}$$

   Of course, we could have seen this using linearity

$$\mathbf{E}(Z) = \mathbf{E}(X+Y) = \mathbf{E}(X) + \mathbf{E}(Y) = \frac{m+1}{2} + \frac{n+1}{2}.$$

   Similarly, by independence,

$$\mathbf{Var}(Z) = \mathbf{Var}(X+Y) = \mathbf{Var}(X) + \mathbf{Var}(Y) = \frac{m^2-1}{12} + \frac{n^2-1}{12}.$$

3. Suppose that there are n dice whose six sides instead of being numbered are labeled by an apple on one side, a banana on two sides and a cherry on three sides. Suppose all n dice are rolled. Let X be the number of apples showing, Y the number of bananas and Z the number of cherries. Find the joint probability $f(x,y)$. Find the marginal probability $f_X(x)$.

   This is the multinomial distribution. Each roll of a die is independent. Also, for each roll, $p = \mathbf{P}(\text{apple}) = \frac{1}{6}$, $q = \mathbf{P}(\text{banana}) = \frac{1}{3}$ and $r = \mathbf{P}(\text{cherry}) = \frac{1}{2}$ such that $p+q+r = 1$. If there are $x \ge 0$ apples, $y \ge 0$ bananas and $z \ge 0$ cherries, such that $x+y+z = n$,

$$f(x,y,z) = \mathbf{P}(X = x \text{ and } Y = y \text{ and } Z = z) = \binom{n}{x,y,z} p^x q^y r^z = \frac{n!}{x!\,y!\,z!}\ \frac{2^y 3^z}{6^n}$$

   The joint probability for $x \ge 0$, $y \ge 0$ and $x+y \le n$ is

$$\begin{aligned} f(x,y) &= \mathbf{P}(X = x \text{ and } Y = y) = f(x,y,n-x-y) \\ &= \frac{n!}{x!\,y!\,(n-x-y)!}\ \left(\frac{1}{6}\right)^x \left(\frac{1}{3}\right)^y \left(\frac{1}{2}\right)^{n-x-y}. \end{aligned}$$

Thus using the binomial formula, the marginal probability

$$f_X(x) = \sum_y f(x,y) = \frac{n!}{x!\,(n-x)!}\left(\frac{1}{6}\right)^x \sum_{y=0}^{n-x} \frac{(n-x)!}{y!\,(n-x-y)!}\left(\frac{1}{3}\right)^y \left(\frac{1}{2}\right)^{n-x-y}$$
$$= \frac{n!}{x!\,(n-x)!}\left(\frac{1}{6}\right)^x \left(\frac{1}{3}+\frac{1}{2}\right)^{n-x} = \binom{n}{x}\left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{n-x}.$$

This is no surprise. If a "success" means an apple is rolled and "failure" means a banana or cherry is rolled, then the number of apples has the binomial disrtribution $X \sim \text{bin}(n, \frac{1}{6})$.

4. *Let X and Y be random variables whose joint probability is $f(1,1) = f(3,2) = .2$, $f(1,2) = .5$, $f(2,2) = .1$ and $f(i,j) = 0$ for other (i,j). Find $f_{X|Y}(x \mid y)$ and $\mathbf{E}(X \mid Y)$.*

   We obtain the marginal probabilities by summing rows of the pmf table using

$$f_X(x) = \sum_y f(x,y), \qquad f_Y(y) = \sum_x f(x,y).$$

	$x=1$	$x=2$	$x=3$	$f_Y(y)$
$y=1$	.2	0	0	.2
$y=2$	.5	.1	.2	.8
$f_X(x)$	.7	.1	.2	1

The conditional probability mass function

$$f_{X|Y}(x\,|\,y) = \mathbf{P}(X = x\,|\,Y = y) = \frac{f(x,y)}{f_Y(y)}$$

so that

$$f_{X|Y}(1\,|\,1) = \frac{f(1,1)}{f_Y(1)} = \frac{.2}{.2} = 1, \qquad f_{X|Y}(2\,|\,1) = \frac{f(2,1)}{f_Y(1)} = 0,$$
$$f_{X|Y}(3\,|\,1) = \frac{f(3,1)}{f_Y(1)} = 0, \qquad f_{X|Y}(1\,|\,2) = \frac{f(1,2)}{f_Y(2)} = \frac{.5}{..8} = .625,$$
$$f_{X|Y}(2\,|\,2) = \frac{f(2,2)}{f_Y(2)} = \frac{.1}{.8} = .125, \qquad f_{X|Y}(3\,|\,2) = \frac{f(3,2)}{f_Y(2)} = \frac{.2}{.8} = .25.$$

The conditional expectation $\mathbf{E}(X\,|\,Y)$ is thus

$$\mathbf{E}(X\,|\,Y=1) = \sum_x x\, f_{X|Y}(x\,|\,1) = (1)(1) + (2)(0) + (3)(0) = 1,$$
$$\mathbf{E}(X\,|\,Y=2) = \sum_x x\, f_{X|Y}(x\,|\,2) = (1)(.625) + (2)(.125) + (3)(.25) = 1.625.$$

5. *Suppose that* $\mathbf{E}(|X|^\alpha) = 0$ *for some* $\alpha > 0$. *Show that* $\mathbf{P}(X = 0) = 1$.

This follows from a version of Chebychov's Inequality. Let $h(x) = |x|^\alpha$ in the basic inequality Theorem 4.6.1. Then for $a = \varepsilon^\alpha > 0$,

$$\mathbf{P}(|X| \ge \varepsilon) = \mathbf{P}(|X|^\alpha \ge \varepsilon^\alpha) \le \frac{\mathbf{E}(|X|^\alpha)}{\varepsilon^\alpha}. \tag{1}$$

Let A_n denote the event that $|X| \ge \frac{1}{n}$. Then the events are monotone $A_n \subset A_{n+1}$ for all n and the event

$$\{X \ne 0\} = \{|X| > 0\} = \bigcup_{n=1}^{\infty} A_n.$$

By the monotone convergence of probability, and (1) with $\varepsilon = 1/n$,

$$\begin{aligned} 0 \le 1 - \mathbf{P}(X = 0) = \mathbf{P}(|X| > 0) = \mathbf{P}\left(\bigcup_{n=1}^{\infty} A_n\right) &= \lim_{n\to\infty} \mathbf{P}(A_n) \\ &\le \lim_{n\to\infty} n^\alpha\, \mathbf{E}(|X|^\alpha) = \lim_{n\to\infty} 0 = 0. \end{aligned}$$

6. *Let* X_n *be a sequence of independent Bernoulli trials* ($X_n \in \{0, 1\}$) *such that* $\mathbf{P}(X_n = 1) = p$ *for all* n. *Show that the sample means converge in probability as* $n \to \infty$,

$$\frac{1}{n}\left(X_1 + \cdots + X_n\right) \xrightarrow{P} p.$$

Let $S_n = X_1 + \cdots + X_n$ as usual and observe that because of independence and the behavior of variance under multiplication by a constant, the random variable S_n/n satisfies

$$\begin{aligned} \mathbf{E}\left(\frac{1}{n} S_n\right) &= \frac{1}{n}\,\mathbf{E}\left(\sum_{j=1}^n X_j\right) = \frac{1}{n}\sum_{j=1}^n \mathbf{E}(X_j) = \frac{1}{n}\sum_{j=1}^n p = p; \\ \mathbf{Var}\left(\frac{1}{n} S_n\right) &= \frac{1}{n^2}\,\mathbf{Var}\left(\sum_{j=1}^n X_j\right) = \frac{1}{n^2}\sum_{j=1}^n \mathbf{Var}(X_j) = \frac{1}{n^2}\sum_{j=1}^n pq = \frac{pq}{n}. \end{aligned}$$

Recall the version of Chebychov's inequality, gotten by applying Theorem 4.6.1 to $h(x) = (x - \mathbf{E}(X))^2$. For all $a > 0$,

$$\mathbf{P}(|X - \mathbf{E}(X)| \ge a) \le \frac{\mathbf{Var}(X)}{a^2}. \tag{2}$$

Recall also the meaning of $Z_n \xrightarrow{P} Z$ "convergence in probability." It means for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}\left(|Z_n - Z| \ge \varepsilon\right) = 0.$$

In our case, $Z_n = S_n/n$, $Z = p$ and apply (2) with $a = \epsilon$, where $\epsilon > 0$ is any positive number. Then

$$\mathbf{P}\left(\left|\frac{1}{n} S_n - p\right| \ge \varepsilon\right) \le \frac{\mathbf{Var}\left(\frac{1}{n} S_n\right)}{\varepsilon^2} = \frac{pq}{\varepsilon^2\, n} \to 0 \text{ as } n \to \infty.$$

Thus the sample average converges in probability to its mean, as to be shown.

7. *Urn 1 contains 10 white and 3 black balls; urn 2 contains 3 white and 5 black balls. Two randomly selected balls are transferred from No. 1 and placed in No. 2 and then 4 balls are taken from the latter. Let the rv X be the number of white balls chosen. Find $f_X(x)$.*

   This is an exercise in the total probability formula. Let us condition on the colors of the balls transferred from urn 1 to urn 2. Three mutually exclusive events are A, that both balls are white, B that one was white and the other was black, and C that both were black. The probabilities are computed as usual for drawing without replacement (*i.e.,* hypergeometric)

$$\mathbf{P}(A) = \frac{\binom{10}{2}\binom{3}{0}}{\binom{13}{2}} = \frac{10\cdot 9}{13\cdot 12} = \frac{15}{26}, \qquad \mathbf{P}(B) = \frac{\binom{10}{1}\binom{3}{1}}{\binom{13}{2}} = \frac{10\cdot 3\cdot 2}{13\cdot 12} = \frac{10}{26},$$

$$\mathbf{P}(C) = \frac{\binom{10}{0}\binom{3}{2}}{\binom{13}{2}} = \frac{3\cdot 2}{13\cdot 12} = \frac{1}{26}.$$

   The conditional pmf's may be computed using the given constitution of the second urn. For $0 \le x \le 4$,

$$f(x\,|\,A) = \frac{\binom{5}{x}\binom{5}{4-x}}{\binom{10}{4}}, \qquad f(x\,|\,B) = \frac{\binom{4}{x}\binom{6}{4-x}}{\binom{10}{4}}, \qquad f(x\,|\,C) = \frac{\binom{3}{x}\binom{7}{4-x}}{\binom{10}{4}}.$$

   Of course we interpret $f(4\,|\,C) = 0$. Using the total probability formula for the pmf,

$$\begin{aligned} f(x) &= \mathbf{P}(A)f(x\,|\,A) + \mathbf{P}(B)f(x\,|\,B) + \mathbf{P}(C)f(x\,|\,C) \\ &= \frac{15}{26\cdot 210}\binom{5}{x}\binom{5}{4-x} + \frac{10}{26\cdot 210}\binom{4}{x}\binom{6}{4-x} + \frac{1}{26\cdot 210}\binom{3}{x}\binom{7}{4-x}. \end{aligned}$$

8. *Roll a die. Let X be the number showing. Then flip X fair coins. Let Y be the number of heads. Find $\mathbf{E}(Y)$.*

   We use the formula $\mathbf{E}(\mathbf{E}(Y \mid X)) = \mathbf{E}(Y)$. The variable X is uniform on $\{1,2,3,4,5,6\}$ so that for integral x such that $1 \le x \le 6$,

$$f_X(x) = \mathbf{P}(X = x) = \frac{1}{6}, \qquad \mathbf{E}(X) = \frac{7}{2}.$$

   We assume that the coins are independent, thus the number of heads in X flips is distributed binomially $Y \sim \text{bin}(X, \frac{1}{2})$. Thus the conditional pmf and expectations are for $1 \le y \le x$,

$$f_{Y|X}(y\,|\,x) = \mathbf{P}(Y = y \mid X = x) = \binom{x}{y} p^y\, q^{x-y} = \binom{x}{y}\frac{1}{2^x}; \qquad \mathbf{E}(Y \mid X = x) = px = \frac{x}{2}.$$

   Thus

$$\mathbf{E}(Y) = \mathbf{E}(\mathbf{E}(Y \mid X)) = \mathbf{E}\left(\frac{X}{2}\right) = \frac{\mathbf{E}(X)}{2} = \frac{7}{4}.$$