

48[4] *An urn contains 100 balls numbered 1 to 100. Four are removed at random without being replaced. Find the probability that the number of the first ball is greater than the number on the last ball.*

Let the sample space be the set of four-tuples showing the numbers of the balls taken in order

$$\Omega = \{(x_1, x_2, x_3, x_4) : 1 \leq x_i \leq 100 \text{ and the } x_i\text{'s are distinct}\}.$$

Let the event that the first ball exceeds the last be

$$A = \{(x_1, x_2, x_3, x_4) \in \Omega : x_1 > x_4\}.$$

Then the complementary set A^c is the set of all outcomes with the last ball greater than the first. Since swapping the order of the balls is a symmetry of Ω , it is equally likely that any pair of numbers occur as the first and the last choice or the last and the first choice. In other words, the symmetry can be given as a mapping $\sigma : \Omega \rightarrow \Omega$ by

$$\sigma((x_1, x_2, x_3, x_4)) = (x_4, x_3, x_2, x_1)$$

which is a one-to-one correspondence of Ω to itself (since it is reversible $\sigma^{-1} = \sigma$). Since σ swaps balls, it maps an outcome where the first ball is greater than the last to an outcome where the last is greater than the first and vice versa. Hence σ is also a one-to-one correspondence $\sigma : A \rightarrow A^c$. It follows that $|A| = |A^c|$ so that

$$\mathbf{P}(A) = \frac{|A|}{|\Omega|} = \frac{|A^c|}{|\Omega|} = \mathbf{P}(A^c)$$

so

$$1 = \mathbf{P}(\Omega) = \mathbf{P}(A \cup A^c) = \mathbf{P}(A) + \mathbf{P}(A^c) = 2\mathbf{P}(A).$$

Thus $\mathbf{P}(A) = \frac{1}{2}$.

48[6] *To start playing a game of chance with a die, it is necessary first to throw a six.*

- (a) *What is the probability that you throw your first six on your third attempt?*
- (b) *What is the probability that you require more than three attempts?*
- (c) *What is the most likely number of attempts before you first throw a six?*
- (d) *After how many throws would your probability of having thrown a six be at least 0.95?*

Let Ω_n denote sample space consisting of all possible ordered sequences of n rolls of a die.

$$\Omega_n = \{(x_1, \dots, x_n) : 1 \leq x_i \leq 6 \text{ for all } i\}.$$

Each roll sequence is equally likely. $|\Omega_n| = 6^n$. The event that the first six occurs on the n th throw

$$A_n = \{(x_1, \dots, x_n) \in \Omega_n : x_i \leq 5 \text{ for } i = 1, 2, \dots, n-1 \text{ and } x_n = 6\}.$$

There are five choices for the first $n-1$ rolls and one for the n th so $|A_n| = 5^{n-1}$. Hence,

$$(a.) \quad \mathbf{P}(A_3) = \frac{|A_3|}{|\Omega_3|} = \frac{5^2}{6^3}, \quad \mathbf{P}(A_n) = \frac{|A_n|}{|\Omega_n|} = \frac{5^{n-1}}{6^n} = \frac{1}{6} \left(\frac{5}{6}\right)^{n-1}.$$

The event that no six occurs in n throws is

$$B_n = \{(x_1, \dots, x_n) \in \Omega_n : x_i \leq 5 \text{ for } i = 1, 2, \dots, n\}.$$

(b.) There are five ways to get a non-six on each roll. Thus the probability that it takes more than three throws to get a six is

$$\mathbf{P}(B_3) = \frac{|B_3|}{|\Omega_3|} = \frac{5^3}{6^3}.$$

(c.) Since $\mathbf{P}(A_n)$ is decreasing with n , the largest $\mathbf{P}(A_n)$ occurs when $n = 1$.

(d.) The event that at least one six is thrown in the first n rolls is B_n^c . Thus we seek n such that $\mathbf{P}(B_n^c) \geq 0.95$ which happens when $n \geq 17$ because

$$\left(\frac{5}{6}\right)^n = \mathbf{P}(B_n) = 1 - \mathbf{P}(B_n^c) \leq 0.05 < \mathbf{P}(B_{n-1}) = \left(\frac{5}{6}\right)^{n-1}$$

happens for $n = 17$ since $0.04507 \approx \mathbf{P}(B_{17}) \leq 0.05 < \mathbf{P}(B_{16}) \approx 0.05409$.

48[9] *A fair coin is tossed four times. What is the probability of*

- (a) *At least three heads?*
- (b) *Exactly three heads?*
- (c) *A run of three or more consecutive heads?*
- (d) *A run of exactly three consecutive heads?*

Let Ω denote sample space consisting of all possible ordered sequences of four tosses

$$\Omega = \{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, HTHH, THTH, TTTH, TTTT\}.$$

All toss sequences are equally likely and $|\Omega| = 2^4 = 16$.

(a.) The event “at least three heads” is $A = \{HHHH, HHHT, HHTH, THHH, HTHH\}$. Thus

$$\mathbf{P}(A) = \frac{|A|}{|\Omega|} = \frac{5}{16}.$$

(b.) The event “exactly three heads” is $B = \{HHHT, HHTH, THHH, HTHH\}$. Thus

$$\mathbf{P}(B) = \frac{|B|}{|\Omega|} = \frac{4}{16}.$$

(c.) The event “a run of three or more consecutive heads” is $C = \{HHHH, HHHT, THHH\}$. Thus

$$\mathbf{P}(C) = \frac{|C|}{|\Omega|} = \frac{3}{16}.$$

(d.) The event “a run of exactly three heads” is $D = \{HHHT, THHH\}$. Thus

$$\mathbf{P}(D) = \frac{|D|}{|\Omega|} = \frac{2}{16}.$$