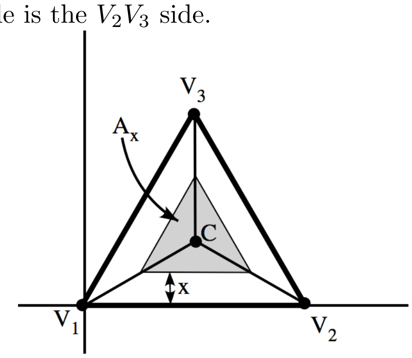
334[14] *A point Q is chosen at random inside an equilateral triangle of unit side. Find the density of the perpendicular distance X to the nearest side of the triangle.*

The probability of a random point being in any set is proportional to the area of the set. Let T be the equilateral triangle. Let h be the largest perpendicular distance that any point of T has to a side, namely, $h = \operatorname{dist}(C, \text{side})$ where C is the center point of T. Let $A_x = \{Q \in T : X(Q) > x\}$ be the subset of points whose perpendicular distance to the nearest side is more that x. Then, for $0 \le x \le h$,

$$1 - F_X(x) = 1 - \mathbf{P}(X \le x) = \mathbf{P}(X > x) = \mathbf{P}(Q \in A_x) = \frac{\operatorname{Area}(A_x)}{\operatorname{Area}(T)}. \tag{1}$$

For convenience, let the vertices of the triangle be located at the points, $V_1 = (0,0)$, $V_2 = (1,0)$ and $V_3 = (\cos\frac{\pi}{3}, \sin\frac{\pi}{3}) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The midpoint is $C = \frac{1}{3}(V_1 + V_2 + V_3) = \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$. Hence, the distance from the center to an edge, say the the bottom edge, is $h = \operatorname{dist}(C, \text{side}) = \frac{\sqrt{3}}{6}$. For points inside the triangle V_1V_2C, the closest side is the bottom edge V_1V_2. For points inside the triangle V_1V_3C, the closest side is the V_1V_3 side. For points inside the V_2V_3C triangle is the V_2V_3 side.



Equilateral triangle T with subtriangle A_x.

The area $\operatorname{Area}(T) = \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} \cdot 1 \cdot \frac{1}{2}\sqrt{3} = \frac{1}{4}\sqrt{3}$. The area of A_x is three times the area of $B_x = A_x \cap \Delta(V_1V_2C)$, the part of A_x within triangle V_1V_2C on the V_1V_2 side of A_x. The height of B_x is $h - x$. The base of B_x decreases linearly from 1 to 0 as x increases from 0 to h thus it is $1 - x/h$. It follows that for $0 \le x \le h$,

$$\operatorname{Area}(A_x) = 3\operatorname{Area}(B_x) = 3 \cdot \frac{1}{2}\operatorname{base}(B_x) \cdot \operatorname{height}(B_x) = \frac{3}{2}\left(1 - \frac{x}{h}\right)(h - x) = \frac{\sqrt{3}}{4}(1 - 2\sqrt{3}x)^2.$$

Using (1), we have computed the cumulative distribution function

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1 - \left(1 - 2\sqrt{3}x\right)^2, & \text{if } 0 \le x \le \frac{1}{6}\sqrt{3}; \\ 1, & \text{if } \frac{1}{6}\sqrt{3} < x. \end{cases}$$

It follows that the distribution function $f_X(x) = F_X'(x)$ for x such that $F_X(x)$ is differentiable. Since F_X is not differentiable only at $x = 0$ and $x = h$, f_X is defined to be zero there. Thus the probability density function is

$$f_X(x) = \begin{cases} 0, & \text{if } x \le 0; \\ 4\sqrt{3}\left(1 - 2\sqrt{3}x\right), & \text{if } 0 < x < \frac{1}{6}\sqrt{3}; \\ 0, & \text{if } \frac{1}{6}\sqrt{3} \le x. \end{cases}$$

A*. *Suppose A and r are positive constants and*

$$f(x) = \begin{cases} \dfrac{c}{x^{r+1}}, & \text{if } x \ge A; \\ \\ 0, & \text{if } x < 0. \end{cases}$$

For what values of the constant c is $f(x)$ a probability density function? Show that the nth moment is finite if and only in $r > n$. Find the expectation and variance in cases they exist.

The function is a density if it is nonnegative (so require $c > 0$) and if the total probability is one. Since $A, r > 0$,

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_A^{\infty} \frac{c\,dx}{x^{r+1}} = \left(-\frac{c}{r}x^{-r}\right)\Big|_A^{\infty} = \frac{c}{rA^r}.$$

Thus if $c = rA^r$ then $f(x)$ is a probability density function. The nth moment about zero exists if and only if $\mathbf{E}(|X|^n)$ is finite.

$$\mathbf{E}(|X|^n) = \int_{-\infty}^{\infty} |x|^n f(x)\,dx = rA^r \int_A^{\infty} \frac{x^n\,dx}{x^{r+1}} = \begin{cases} \left(rA^r x^{n-r}\right)\big|_A^{\infty}, & \text{if } n \ne r; \\ rA^r(\ln x)|_A^{\infty}, & \text{if } r = n. \end{cases}$$

is finite if and only if $n < r$.

Thus if $r > 1$, the first moment, so the expectation exists, and

$$\mathbf{E}(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx = rA^r \int_A^{\infty} \frac{x\,dx}{x^{r+1}} = \left(\frac{rA^r x^{1-r}}{1-r}\right)\Bigg|_A^{\infty} = \frac{rA}{r-1}.$$

If $r > 2$, the second moment exists and

$$\mathbf{E}(X^2) = \int_{-\infty}^{\infty} x^2\,f(x)\,dx = rA^r \int_A^{\infty} \frac{x^2\,dx}{x^{r+1}} = \left(\frac{rA^r x^{2-r}}{2-r}\right)\Bigg|_A^{\infty} = \frac{rA^2}{r-2}.$$

It follows that the variance

$$\mathbf{Var}(X) = \mathbf{E}(X^2) - \mathbf{E}(X)^2 = \frac{rA^2}{r-2} - \frac{r^2A^2}{(r-1)^2} = \frac{rA^2}{(r-2)(r-1)^2}.$$

B*. *Let X have the χ^2 distribution with parameter n. Show that $Y = \sqrt{X/n}$ has the χ distribution with parameter n, which means that Y has the probability density function*

$$f_Y(y) = \begin{cases} \dfrac{2\left(\frac{n}{2}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} y^{n-1} e^{-\frac{n}{2}y^2}, & \text{if } y > 0; \\ \\ 0, & \text{if } y \le 0. \end{cases}$$

The χ^2 distribution has the probability density function

$$f_X(x) = \begin{cases} \dfrac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}, & \text{if } x > 0; \\ \\ 0, & \text{if } x \le 0. \end{cases}$$

Let $F_X(x) = \mathbf{P}(X \le x)$ be the cumulative distribution function. To find the distribution function for Y, for $y \ge 0$,

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}\left(\sqrt{\frac{X}{n}} \le y\right) = \mathbf{P}\left(X \le ny^2\right) = F_X\left(ny^2\right)$$

since $X \ge 0$. $F_X(x)$ is differentiable for $x > 0$ and $F_X(0) = 0$ so for $y > 0$,

$$\begin{aligned} f_Y(y) &= F_Y'(y) = F_X'(ny^2)\, 2ny = f_X(ny^2)\, 2ny \\ &= \frac{n^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\,\Gamma\left(\frac{n}{2}\right)}\, y^{n-2}\, e^{-\frac{n}{2}y^2} \cdot 2ny = \frac{2\left(\frac{n}{2}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\, y^{n-1}\, e^{-\frac{n}{2}y^2}. \end{aligned}$$

If $y < 0$, then since $X \ge 0$,

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}\left(\sqrt{\frac{X}{n}} \le y\right) = 0$$

so $f_Y(y) = 0$ for $y \le 0$. Thus $f_Y(y)$ has the χ distribution with parameter n.