Math 5010 § 1. **Solutions to Second Homework**
Treibergs January 31, 2009

48[15] *Four fair dice are rolled and the four numbers are multiplied together, What is the probability that this product*

(a) Is divisible by 5?

(b) Has last digit 5?

(a.) The product is divisible by 5 if ands only if at least one of its factors is 5. Let

$$\Omega = \{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 : 1 \le x_j \le 6 \text{ for all } j\}$$

be the sample space of all possible outcomes of four rolls, each of which is equally likely, and $|\Omega| = 6^4$. Let

$$A = \{(x_1, x_2, x_3, x_4) \in \Omega : \ x_i = 5 \text{ for some } i \ \}$$

be the event that a roll is 5, equivalently that the product is divisible by 5. Then A^c is the event that none of the rolls is a five. There are five ways to roll a non-five so that

$$\mathbf{P}(A) = 1 - \mathbf{P}(A^c) = 1 - \frac{|A^c|}{|\Omega|} = 1 - \frac{5^4}{6^4} \approx 0.5178.$$

(b.) The product ends in five if and only if 5 is a factor but 2 is not a factor, which is equivalent to none of the rolls are even and at least one of them is 5. Thus, the event the product ends in five is

$$B = \{(x_1, x_2, x_3, x_4) \in \Omega : \ x_i \in \{1, 3, 5\} \text{ for all } i \text{ but } x_i = 5 \text{ for some } i \ \} = E \setminus F$$

where E is the event that all rolls are odd

$$E = \{(x_1, x_2, x_3, x_4) \in \Omega : \ x_i \in \{1, 3, 5\} \text{ for all } i \ \}$$

and F is the event that all rolls are odd but none is a five

$$F = \{(x_1, x_2, x_3, x_4) \in \Omega : \ x_i \in \{1, 3\} \text{ for all } i \ \}.$$

It follows that

$$\mathbf{P}(B) = \frac{|B|}{|\Omega|} = \frac{|E| - |F|}{\Omega|} = \frac{3^4 - 2^4}{6^4} \approx 0.0502.$$

48[22] *When are the following true?*

(a) $A \cup (B \cap C) = (A \cup C) \cap (A \cup C)$.

(b) $A \cap (B \cap C) = (A \cap B) \cap C$.

(c) $A \cup (B \cup C) = A \setminus (B \setminus C)$.

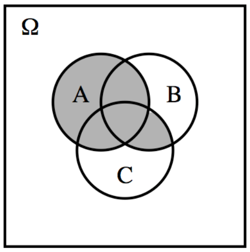
(d) $A \setminus (B \setminus C) = (A \setminus B) \setminus C$.

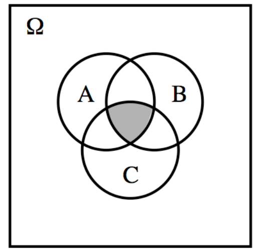
(e) $A \,\Delta\, (B \,\Delta\, C) = (A \,\Delta\, B) \,\Delta\, C$.

(f) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

(g) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

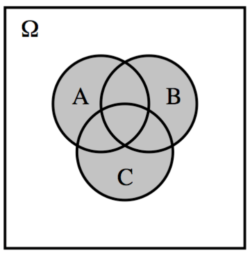
We indicate the named sets by shading in the corresponding Venn diagrams. If the Venn diagrams are the same for both sets, then the equality holds always. If there are different regions shaded in the diagram, then equality implies that the regions that are unshaded in one diagram and shaded in the other should be empty.

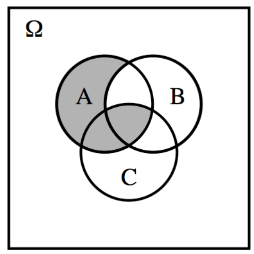




(a.) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ holds always.

(b.) $A \cap (B \cap C) = (A \cap B) \cap C$ holds always.





(c.) $A \cup (B \cup C)$

$A \setminus (B \setminus C)$

Equality holds if all four conditions hold:

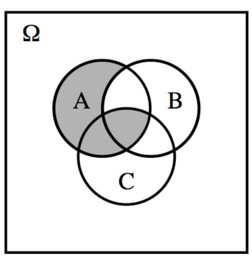
$$A \cap B \cap C^c = \emptyset, \quad A^c \cap B \cap C^c = \emptyset, \quad A^c \cap B \cap C = \emptyset, \quad A^c \cap B^c \cap C = \emptyset.$$

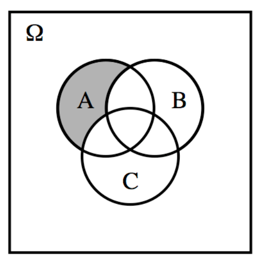
Equivalently, using $E \cap F = \emptyset$ and $E \cap F^c = \emptyset$ if and only if $E = \emptyset$, the first two and the last two say

$$B \cap C^c = \emptyset, \qquad A^c \cap C = \emptyset$$

which are equivalent to

$$B \subseteq C \subseteq A.$$





(d.) $A \setminus (B \setminus C)$

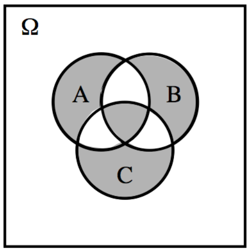
$(A \setminus B) \setminus C$

Equality holds if the two conditions hold:

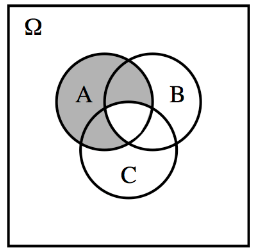
$$A \cap B \cap C = \emptyset, \qquad A \cap B^c \cap C = \emptyset.$$

Equivalently,

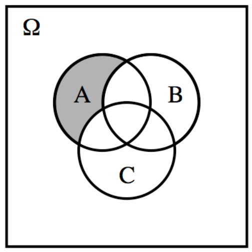
$$A \cap C = \emptyset.$$



(e.) $A \,\Delta\, (B \,\Delta\, C) = (A \,\Delta\, B) \,\Delta\, C$
holds always.



(f.) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$
holds always.



(g.) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
holds always.

48[23] *If M students born in 1985 are attending a lecture, find that the probability that at least two of them share a birthday. Show that if $m \ge 23$ then $\mathbf{P} > \frac{1}{2}$. What difference would it make if they were born in 1988?*

Let Ω_m be the sample space consisting of m-tuples of birthdays given as a number from 1 to 365. ($\mathbb{Z}$ is the usual symbol for the integers.)

$$\Omega = \{(x_1, x_2, \dots, x_m) \in \mathbb{Z}^m : 1 \le x_j \le 365 \text{ for all } j\}$$

Every list of birthdays is equally likely and $|\Omega| = 365^m$. Let A_m denote the event that at least two people share a birthday

$$A_m = \{(x_1, x_2, \dots, x_m) \in \Omega : \; x_i = x_j \text{ for some } i \ne j \;\}$$

The complementary event is that noone shares a birthday, hence there are 365 birthday possibilities for the first person, $365 - 1$ birthdays other than the first persons for the second, 365-2 birthdays other than the first two persons birthdays and so on:

$$|A_m^c| = 365 \cdot 364 \cdot \dots \cdot (365 - m + 1) = \tfrac{365!}{(365-m)!}$$

This quantity is understood to be zero when $m > 365$ since it is impossible to assign distinct birthdays in that case. Thus the probability is

$$\mathbf{P}(A_m) = 1 - \mathbf{P}(A_m^c) = 1 - \tfrac{|A^c|}{|\Omega|} = 1 - \tfrac{365!}{(365-m)!\,365^m}$$

which is 1 if $m > 365$. Observe that this increases with m. Computing $\mathbf{P}(A_{22}) \approx 0.476$, $\mathbf{P}(A_{23}) \approx 0.507$ and $\mathbf{P}(A_{24}) \approx 0.538$ so that $\mathbf{P}(A_m) > \frac{1}{2}$ iff $m \ge 23$.

If all the students were born in 1988, a leap year, then there would be 366 different birthdays in the year so the probability is now

$$\mathbf{P}(A_m) = 1 - \tfrac{366!}{(366-m)!\,366^m}$$

which is 1 if $m > 366$. Observe that this increases with m but has slightly lower values. Computing $\mathbf{P}(A_{22}) \approx 0.475$, $\mathbf{P}(A_{23}) \approx 0.506$ and $\mathbf{P}(A_{24}) \approx 0.537$ so that still, $\mathbf{P}(A_m) > \frac{1}{2}$ iff $m \ge 23$. (My calculator couldn't handle 356!. I had to write a program to compute $\prod_{j=1}^m \left(\frac{366-j}{365}\right)$.)