76[1]  *The probability that an archer hits the target when it is windy is 0.4; when it is not windy,
her probability of hitting the target is 0.7. On any shot, the probability of a gust of wind is
0.3. Find the probability that*

  (a)  *On a given shot there is a gust of wind and she hits the target.*

  (b)  *She hits the target with her first shot.*

  (c)  *She hits the target exactly once in two shots.*

  (d)  *There was no gust of wind on an occasion when she missed.*

Let $A$ denote the event that she hits the target on any given try and $B$ be the event that
there is a gust of wind on a given try. We are given $\mathbf{P}(A|B) = 0.4$, $\mathbf{P}(A|B^c) = 0.7$ and
$\mathbf{P}(B) = 0.3$.

(a.)  $\mathbf{P}(A \cap B) = \mathbf{P}(B)\,\mathbf{P}(A|B) = (0.3)(0.4) = 0.12.$

(b.)  Using $B$ to partition $A$ we get

$$\mathbf{P}(A) = \mathbf{P}(A \cap B) + \mathbf{P}(A \cap B^c) = \mathbf{P}(B)\,\mathbf{P}(A|B) + \mathbf{P}(B^c)\,\mathbf{P}(A|B^c)$$
$$= (0.3)(0.4) + (1 - 0.3)(0.7) = 0.12 + 0.49 = 0.61.$$

(c.)  Let $A_i$ denote the event that she hits the target on her $i$th try. We make the assumption
that $A_1$ and $A_2$ are independent events and each has the probability given in part (b.).
Then the event $C$ of hitting the target exactly once in two tries is the same event as either
hitting on the first try and missing on the second or missing on the first try and hitting
on the second:  $C = (A_1 \cap A_2^c) \cup (A_1^c \cap A_2)$. These are disjoint events so that we may
add probabilities. Since $A_1$ and $A_2$ are independent, we can multiply to get probability of
intersection.

$$\mathbf{P}(C) = \mathbf{P}(A_1 \cap A_2^c) + \mathbf{P}(A_1^c \cap A_2)$$
$$= \mathbf{P}(A_1)\,\mathbf{P}(A_2^c) + \mathbf{P}(A_1^c)\,\mathbf{P}(A_2)$$
$$= (0.61)(1 - 0.61) + (1 - 0.61)(0.61) = 0.4758.$$

(d.)  Using Bayes' formula (or part (b.)),

$$\mathbf{P}(B^c|A^c) = \frac{\mathbf{P}(A^c \cap B^c)}{\mathbf{P}(A^c)} = \frac{\mathbf{P}(A^c \cap B^c)}{\mathbf{P}(A^c \cap B) + \mathbf{P}(A^c \cap B^c)}$$
$$= \frac{\mathbf{P}(B^c)\,\mathbf{P}(A^c|B^c)}{\mathbf{P}(B)\,\mathbf{P}(A^c|B) + \mathbf{P}(B^c)\,\mathbf{P}(A^c|B^c)}$$
$$= \frac{(1 - 0.3)(1 - 0.7)}{(0.3)(1 - 0.4) + (1 - 0.3)(1 - 0.7)} = \frac{0.21}{0.18 + 0.21} = \frac{7}{13}.$$

76[5]  *Candidates are allowed at most three attempts at a given test. Given $j - 1$ previous failures,
the probability that the candidate fails on his $j$th attempt is $p_j$. If $p_1 = 0.6$, $p_2 = 0.4$ and
$p_3 = 0.75$, find the probability that the candidate:*

  (a)  *Passes at the second attempt.*

  (b)  *Passes at the third attempt.*

  (c)  *Passes given that he failed on the first attempt.*

*(d) Passes at the second attempt, given that he passes.*

The interpretation of this question requires that you use your experience. We shall assume that the candidate takes tests until he passes or until he fails his third try, and does not take a test again after he has passed. Let $F_i$ denote the event that the candidate fails the $i$th test. Then we are given the conditional prbabilities $p_1 = \mathbf{P}(F_1)$, $p_2 = \mathbf{P}(F_2|F_1)$ and $p_3 = \mathbf{P}(F_3|F_1 \cap F_2)$.

(a.) The event that he passes on the second try means that in addition he failed on the first try, so

$$\mathbf{P}(F_1 \cap F_2^c) = \mathbf{P}(F_1)\,\mathbf{P}(F_2^c|F_1) = p_1(1-p_2) = (0.6)(0.6) = 0.36.$$

(b.) The event that he passes on the third try means that in addition he failed on the first two tries, so

$$\mathbf{P}(F_1 \cap F_2 \cap F_3^c) = \mathbf{P}(F_1)\,\mathbf{P}(F_2|F_1)\,\mathbf{P}(F_3^c|F_1 \cap F_2) = p_1 p_2(1-p_3) = (0.6)(0.4)(0.25) = 0.06.$$

(c.) The event that he passes is the event that he passes on his first or on his second or on his third try. Thus, using (a.) and (b.), and the fact that the event of passing on the second try is disjoint from the event of passing on the third try,

$$
\begin{aligned}
\mathbf{P}(F_1^c \cup (F_1 \cap F_2^c) \cup (F_1 \cap F_2 \cap F_3^c)|F_1) &= \frac{\mathbf{P}\big(F_1 \cap [F_1^c \cup (F_1 \cap F_2^c) \cup (F_1 \cap F_2 \cap F_3^c)]\big)}{\mathbf{P}(F_1)} \\
&= \frac{\mathbf{P}\big((F_1 \cap F_2^c) \cup (F_1 \cap F_2 \cap F_3^c)\big)}{\mathbf{P}(F_1)} \\
&= \frac{\mathbf{P}(F_1 \cap F_2^c) + \mathbf{P}(F_1 \cap F_2 \cap F_3^c)}{\mathbf{P}(F_1)} \\
&= \frac{0.36 + 0.06}{0.6} = \frac{0.42}{0.6} = 0.7.
\end{aligned}
$$

Note that we have used the set identity $E \cap (F \cup G \cup H) = (E \cap F) \cup (E \cap G) \cup (E \cap H)$.
(d.) Using (a.) and (b.),

$$
\begin{aligned}
&\mathbf{P}(F_1 \cap F_2^c|F_1^c \cup (F_1 \cap F_2^c) \cup (F_1 \cap F_2 \cap F_3^c)) \\
&= \frac{\mathbf{P}\big((F_1 \cap F_2^c) \cap [F_1^c \cup (F_1 \cap F_2^c) \cup (F_1 \cap F_2 \cap F_3^c)]\big)}{\mathbf{P}\big(F_1^c \cup (F_1 \cap F_2^c) \cup (F_1 \cap F_2 \cap F_3^c)\big)} \\
&= \frac{\mathbf{P}(F_1 \cap F_2^c)}{\mathbf{P}\big(F_1^c\big) + \mathbf{P}(F_1 \cap F_2^c) + \mathbf{P}(F_1 \cap F_2 \cap F_3^c)} \\
&= \frac{0.36}{0.4 + 0.36 + 0.06} = \frac{0.36}{0.82} = \frac{18}{41}.
\end{aligned}
$$

76[9] *A 12-sided die A has 9 green faces and 3 white faces, whereas another 12-sided die B has 3 green faces and 9 white faces. A fair coin is tossed once. If it falls heads, a series of throws is made with die A alone; if it falls tails then only the die B is used.*

(a) *Show that the probability that green comes up on the first throw is $\frac{1}{2}$.*

(b) *If green turns up at the first throw, what is the probability that die A is being used?*

(c) *Given that green turns up at the first two throws, what is the probability that green turns up at the third throw?*

Let $H$ denote the event that the coin lands heads which is equivalent to the event that die $A$ is used. Let $G_i$ denote the event that green turns up on the $i$th throw. Since rolling a green depends on which die is used, the probability of rolling green is conditioned on the result of the coin flip, namely, $\mathbf{P}(G_1|A) = \frac{3}{4}$ and $\mathbf{P}(G_1|A^c) = \frac{1}{4}$.

(a.) Using the partitioning formula,

$$
\begin{aligned}
\mathbf{P}(G_1) &= \mathbf{P}\big((G_1 \cap A) \cup (G_1 \cap A^c)\big) \\
&= \mathbf{P}(G_1 \cap A) + \mathbf{P}(G_1 \cap A^c) \\
&= \mathbf{P}(G_1|A)\,\mathbf{P}(A) + \mathbf{P}(G_1|A^c)\,\mathbf{P}(A^c) \\
&= \frac{3}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{2}.
\end{aligned}
$$

(b.) Using (a.),

$$
\begin{aligned}
\mathbf{P}(A|G_1) &= \frac{\mathbf{P}(A \cap G_1)}{\mathbf{P}(G_1)} \\
&= \frac{\mathbf{P}(G_1|A)\,\mathbf{P}(A)}{\mathbf{P}(G_1)} \\
&= \frac{\frac{3}{4} \cdot \frac{1}{2}}{\frac{1}{2}} = \frac{3}{4}.
\end{aligned}
$$

(c.) Since a single die is used for a series of throws, we may assume that the events $G_i$ are conditionally independent and with equal probability, namely, for $i \neq j$,

$$
\mathbf{P}(G_i \cap G_j|A) = \mathbf{P}(G_i|A) \cdot \mathbf{P}(G_j|A) = \left(\frac{3}{4}\right)^2.
$$

Similarly for intersecting more events or conditioning on $A^c$. Thus, applying the partitioning formula to the numerator and denominator, and using conditional independence,

$$
\begin{aligned}
\mathbf{P}(G_3|G_1 \cap G_2) &= \frac{\mathbf{P}(G_1 \cap G_2 \cap G_3)}{\mathbf{P}(G_1 \cap G_2)} \\
&= \frac{\mathbf{P}\big((G_1 \cap G_2 \cap G_3 \cap A) \cup (G_1 \cap G_2 \cap G_3 \cap A^c)\big)}{\mathbf{P}\big((G_1 \cap G_2 \cap A) \cup (G_1 \cap G_2 \cap A^c)\big)} \\
&= \frac{\mathbf{P}(G_1 \cap G_2 \cap G_3 \cap A) + \mathbf{P}(G_1 \cap G_2 \cap G_3 \cap A^c)}{\mathbf{P}(G_1 \cap G_2 \cap A) + \mathbf{P}(G_1 \cap G_2 \cap A^c)} \\
&= \frac{\mathbf{P}(G_1 \cap G_2 \cap G_3|A)\,\mathbf{P}(A) + \mathbf{P}(G_1 \cap G_2 \cap G_3|A^c)\,\mathbf{P}(A^c)}{\mathbf{P}(G_1 \cap G_2|A)\,\mathbf{P}(A) + \mathbf{P}(G_1 \cap G_2|A^c)\,\mathbf{P}(A^c)} \\
&= \frac{\mathbf{P}(G_1|A)^3\,\mathbf{P}(A) + \mathbf{P}(G_1|A^c)^3\,\mathbf{P}(A^c)}{\mathbf{P}(G_1|A)^2\,\mathbf{P}(A) + \mathbf{P}(G_1|A^c)^2\,\mathbf{P}(A^c)} \\
&= \frac{\left(\frac{3}{4}\right)^3 \cdot \frac{1}{2} + \left(\frac{1}{4}\right)^3 \cdot \frac{1}{2}}{\left(\frac{3}{4}\right)^2 \cdot \frac{1}{2} + \left(\frac{1}{4}\right)^2 \cdot \frac{1}{2}} = \frac{27 + 1}{4 \cdot (9 + 1)} = \frac{7}{10}.
\end{aligned}
$$