Math 5210 Midterm 1 - 2/11/15 Name

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1. Define the following terms:

   (a) (10 pts) *Metric space.*

   (b) (10 pts) *Complete* metric space (Give a detailed definition, not just one line).

   (c) (10 pts) *Normed vector space* (over $\mathbb{R}$).

(d) (15 pts) If X, Y are metric spaces and $f : X \to Y$ a map, define:

i. f is *continuous*.

ii. f is *uniformly continuous*.

iii. f is *Lipschitz*.

2. Explain why the following facts are true:

(a) (5 pts) If $f : X \to Y$ is Lipschitz, then f is uniformly continuous.

(b) (10 pts) A normed vector space is a metric space.

(c) (10 pts) $\mathbb{R}^2$ with norm $||(x,y)||_\infty = \max\{|x|,|y|\}$ is a complete metric space. You may use the completeness of $\mathbb{R}$.

3. Let $\phi : \mathbb{R} \to \mathbb{R}$.

   (a) (5 pts) Define what it means for ϕ to be a *convex function.*

   (b) (10 pts) State Jensen's inequality.

(c) (10 pts) Apply Jensen's inequality to derive the inequality between geometric and arithmetic means. This is the statement that for any $a_1, a_2, \ldots, a_n > 0$,

$$(a_1 a_2 \ldots a_n)^{\frac{1}{n}} \leq \frac{a_1 + a_2 + \ldots + a_n}{n} \qquad (1)$$

(d) (5 pts) State and prove the necessary and sufficient condition for equality in (1) in the case $n = 2$.