1. Let X be a metric space and let $f : X \to X$.

   (a) (10 pts) Define what it means for f to be a *contraction*

   (b) (10 pts) State the contraction mapping theorem.

   (c) (20 pts) Sketch a proof of the contraction mapping theorem.

2. (10 pts) Let X be a compact metric space, and let $(C(X), d_\infty)$ be the space of continuous $\mathbb{R}$-valued functions on X with metric $d_\infty(f, g) = max\{|f(x) - g(x)| : x \in X\}$. Show that a sequence $\{f_n\}$ in $C(X)$ converges to f in $C(X), d_\infty$ if and only if $f_n \to f$ *uniformly* on X.

3. (10 pts) Let $U \subset \mathbb{R}^m$ be open, let $f : U \to \mathbb{R}^n$, and let $p \in U$.

   (a) Define what it means for f to be *differentiable* at p.

   (b) (10 pts) Suppose that there exists a continuous map $A : U \times U \to L(\mathbb{R}^m, \mathbb{R}^n)$ so that for all $y \in U$, $f(y) - f(x) = A(x, y)(y - x)$. Prove that f is differentiable at every $p \in U$ and that $d_p f = A(p, p)$.

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4. (15 pts) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function, and consider the initial value problem for the first order differential equation

$$\frac{dx}{dt} = f(t, x(t)), \quad x(0) = x_0, \tag{1}$$

for a function $x(t)$ defiined in an interval $(-a, a)$ for some $a > 0$.

(a) Derive an integral equation that is equivalent to (2), and that is an equation for fixed points.

(b) (15 pts) Look at the special case of (2) where $f(t, x) = x$ and $x_0 = 1$:

$$\frac{dx}{dt} = x(t), \quad x(0) = 1, \tag{2}$$

Apply 3 times the iteration procedure of the proof ot the Contraction Mapping Theorem starting from $x = 1$ write down what you get, and explain why it is reasonable.