Math 5210 § 1. **Midterm Exam** Name:____________________
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All spaces are metric spaces.
Do all problems, but you have a choice on # 4.
Exam is closed book and closed note. Good Luck!

1. (a) [10] Let $\{x_m\} \subset X$. What does it mean for $x_n \to x$ as $n \to \infty$?

   (b) [10] If $x_n \to x$ and $y_n \to y$ in the metric space (X, d), prove that $d(x_n, y_n) \to d(x, y)$ in $\mathbb{R}$.

2. (a) [10] If X is compact and $f : X \to Y$ is continuous, prove $f(X)$ is compact.

   (b) [10] If X is compact and $f : X \to Y$ is continuous, prove that f is uniformly continuous.

3. (a) [10] Using the definition of the Riemann Integral (and its existence), prove that if $f : [a, b] \to X_{\text{Banach}}$ is continuous, then

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

   (b) [10] Let $f_n, f : [a, b] \to X_{\text{Banach}}$, f_n are continuous and $f_n \to f$ uniformly, prove

$$\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx.$$

   (c) [10] Suppose that the convergence in (3b) is pointwise, but not uniform. Give a counterexample to show that the integrals may not converge in this case.

4. [30] Pick ONE of the following theorems to carefully prove (and state if required.)

   (a) State and prove the Contraction Mapping Theorem. Include the definition of a contraction mapping.

   (b) State and prove the chain rule for compositions of differentiable maps between Banach Spaces. Include the definition of what it means for $f : X \to Y$ to be differentiable at $x \in X$.

   (c) Let X be compact and Y be complete. Consider the function metric space

$$\mathcal{C}(X, Y) = \{f : X \to Y : f \text{ is continuous}\}, \qquad d_\infty(f, g) = \sup_{x \in X} d(f(x), g(x))$$

   We've shown that $\mathcal{C}(X, Y)$ is a metric space. Show that $\mathcal{C}(X, Y)$ is complete.