MATH 5210-001

EXAM 2

Instructions. There are 5 problems, each worth the same number of points. Do one of problems 1 and 2, and two of problems 3,4 and 5. Justify your answers. The exam is closed bookend notes.

1a. Let $I = [0, 1]$, and state what it means for the sequence $\{f_n\} \subset C(I, \mathbf{R})$ to be equicontinuous on I.

b. State the Arzela-Ascoli Theorem.

c. Is the sequence $\{ne^{\sin(x/n)}\}$ equicontinuous on I? Does it have a convergent subsequence?

2a. Define what it means for a subset A to be dense in a metric space X.

b. State the Stone-Weierstrass Theorem (for real valued functions).

c. Let $\mathcal{E}$ denote the set of polynomials in one real variable (with real coefficients) which only have terms of even degree. Is $\mathcal{E}$ dense in $C([0,1])$? Is $\mathcal{E}$ dense in $C([-1,1])$?

3a. State the definition of the Lebesgue outer measure $\mu^*(E)$ of a set $E \subset \mathbf{R}$.

b. State the definition of a measurable set (with respect to the Lebesgue outer measure).

c. Show that if $\mu^*(E) = 0$, then E is measurable.

4a. State the Monotone Convergence Theorem (assume the domain is a measurable set in $\mathbf{R}$ with respect to the Lebesgue measure).

b. Suppose that $E \subset \mathbf{R}$ is measurable, and that $f_n : E \to \mathbf{R}$ is measurable with $f_n \geq f_{n+1} \geq 0$ for all $n = 1, 2, \ldots$. Show by way of an example, that it is not necessarily true that $\lim \int_E f_n = \int_E \lim f_n$.

c. If in part b, it is assumed that f_1 was integrable on E, then show that $\lim \int_E f_n = \int_E \lim f_n$.

5a. For a measure space $(X, \mathcal{M}, \mu)$, and $1 \leq p < \infty$, define $L^p(X)$, and the norm $\| \cdot \|_p$.

b. State Holder's Inequality.

c. If X has finite measure, show that if $f \in L^2(X)$, then $f \in L^1(X)$.