MATH 5210-001

FINAL EXAM

Instructions. There are 7 problems, each worth the same number of points. Do 6 out of the 7. Justify your answers.

1a. State the definition of a norm on a vector space V, and the definition of two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ being equivalent on V.

b. Show that the norms $\|\cdot\|_\infty$ and $\|\cdot\|_2$ (the Euclidean norm) are equivalent on $\mathbf{R}^n$.

c. Show that on $C([0,1])$, the sup norm $\|\cdot\|_\infty$ and the L^1 norm, $\|\cdot\|_1$ are not equivalent.

2. State the definition of a set $K \subset X$ being compact, where X is a metric space. Are the following sets compact? Give a proof of your answer (you may use theorems proven in class).

a. $\overline{B}_1(0) \subset \mathbf{R}^n$.

b. $\overline{B}_1(0) \subset \ell^2$.

c. $\overline{A} \subset L^2([0,\pi])$ where $A = \{\sin(nx) : n = 1, 2, \dots\}$.

d. $\overline{A} \subset C([0,\pi])$ where $A = \{\sin(\frac{x}{n}) : n = 1, 2, \dots\}$.

3a. State the definition of the Lebesgue outer measure $\mu^*(E)$ of a set $E \subset \mathbf{R}$.

b. State the definition of a measurable set (with respect to the Lebesgue outer measure).

c. Show that if $\mu^*(E) = 0$, then E is measurable.

d. Show that if $E \subset \mathbf{R}$ is countable, then $\mu^*(E) = 0$.

4a. Define what it means for a subset A to be dense in a metric space X.

b. State the Stone-Weierstrass Theorem.

c. Let $\mathcal{E}$ denote the set of polynomials in one real variable (with real coefficients) which only have terms of even degree. Is $\mathcal{E}$ dense in $C([0,1])$? Is $\mathcal{E}$ dense in $C([-1,1])$?

5a. State the definition of $F : X \to Y$ is uniformly continuous on $A \subset X$, where X and Y are metric spaces.

b. Show that the map $T : L^2([a,b]) \to \mathbf{R}$ is uniformly continuous on $L^2([a,b])$ where $[a,b]$ is a compact interval and

$$T(f) = \int_a^b f(x)\, dx$$

c. Show that $T : C([a,b]) \to \mathbf{R}$ is uniformly continuous on $C([a,b])$ where T is defined as above.

6a. State the Monotone Convergence Theorem (assume the domain is a measurable set in $\mathbf{R}$ with respect to the Lebesgue measure).

b. Suppose that $E \subset \mathbf{R}$ is measurable, and that $f_n : E \to \mathbf{R}$ is measurable with $f_n \geq f_{n+1}$ for all $n = 1, 2, \ldots$. Is it necessarily true that $\lim \int_E f_n = \int_E (\lim f_n)$? Either give a proof (you may use theorems from class) or give a counterexample.

7a. Let $I = [0, 1]$, and state what it means for the sequence $\{f_n\} \subset C(I)$ to be equicontinuous on I.

b. State the Arzela-Ascoli Theorem.

c. Is the sequence $\{ne^{\sin(x/n)}\}$ equicontinuous on I? Does it have a convergent subsequence?