Math 5210 § 2. **Second Midterm Exam** Name: Solutions
Treibergs Feb. 26, 2025

1. *Let (M,d) be a metric space and let $E,F \subset M$. Define: E is open. Define F is closed. Define $x \in M$ is a limit point of F. Prove directly that F is closed if and only if it contains its limit points.*

   E is open if for every $x \in E$ there is $r > 0$ so that $B_r(x) \subset E$, where $B_r(x) = \{y \in M : d(x,y) < r\}$ is the open ball of radius r about z. F is closed if the complememt $F^c = M \backslash F$ is open.

   $x \in M$ is a limit point of F if for every $\varepsilon > 0$ we have $(B_\varepsilon(x)\backslash\{x\}) \cap F \neq \emptyset$.

   Assume that F is closed. Arguing by contrapositive, we show if x is not in F then it is not a limit point of F. Since $x \in F^c$ is open, there is an $r > 0$ such that $B_r(c) \subset F^c$. Now if $\varepsilon < r$ then $B_\varepsilon(x)\backslash\{x\} \subset B_r(x) \subset F^c$ so $(B_\varepsilon(x)\backslash\{x\}) \cap F = \emptyset$ so x is not a limit point of F.

   Now, assume that F contains its limit points. Arguing by contrapositive, we show if F is not closed there is a limit point of F not in F. Thus F^c is not open: there is $x \in F^c$ such that for all $\varepsilon > 0$ the ball $B_\varepsilon(x) \cap F \neq \emptyset$. As $x \notin F$ this says x is a limit point of F which is not in F.

2. *Let (M,d) be a metric space. Define: M is* compact. *Complete the statement of a theorem you'll use to answer the question.*

   Theorem. (M,d) *is a compact metric space if and only if ...*

   Let (M,d) be a compact metric space. Suppose that (F_n) is a decreasing sequence of nonempty closed sets in M, and that $\bigcap_{n=1}^\infty F_n$ is contained in an open set G. Show that $F_n \subset G$ for all but finitely many n.

   Our author defines (M,d) to be compact if it is complete and totally bounded.

   Theorem. (M,d) is a compact metric space if and only if every open cover has a finite subcover. Namely, if $\{G_\alpha\}_{\alpha\in A}$ is a collection of open sets such that $M \subset \bigcup_{\alpha\in A} G_\alpha$ then there are finitely many indices $\{\alpha_1,\ldots,\alpha_k\}$ such that $M \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_k}$.

   To prove the assertion, let $\mathcal{I} = \bigcap_{n=1}^\infty F_n$ be the intersection. Consider the collection of open sets $\{G\} \cup \{F_n^c : n \in \mathbb{N}\}$. This collection covers M. If $x \in \mathcal{I}$ then $x \in G$. If $x \notin \mathcal{I}$ then there is n_0 so that $x \notin F_{n_0}$, in other words $x \in F_{n_0}^c$.

   By the Theorem. there is a finite subcollection $n_1 < n_2 < \cdots < n_k$ such that

   $$M \subset G \cup F_{n_1}^c \cup \cdots \cup F_{n_k}^c.$$

   It follows that $F_j \subset G$ for all j except possibly for $j \le n_k$. This is because of nesting: if $\ell > n_k \ge j$ then $F_\ell \subset F_{n_k} \subset F_j$ so that $F_\ell \cap F_j^c = \emptyset$ for all $j = 1,\ldots,n_k$, thus $F_\ell \subset G$.

   Another argument may be given using the sequential characterization of compactness. If the conclusion fails, there is a subsequence $x_{n_k} \in F_{n_k}\backslash G$ for $k = 1,2,\ldots,\infty$. By compactness there is a convergent subsequence of (x_{n_k}) converging to $x \in M$. Since $(x_{n_k}) \subset G^c$, by the sequential characterization of closedness, $x \in G^c$. On the other hand, by nesting, for $n_j \ge m$ we have $x_{n_j} \in F_{n_j} \subset F_m$ so the characterization of closedness, we have $x \in F_m$ for all m. Thus $x \in \mathcal{I} \subset G$, a contradiction,

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) Statement: *Suppose the map on metric spaces $f : (M,d) \to (N,\rho)$ is continuous. If $f(M)$ is connected then M is connected.*

   False. Let $f : \mathbf{R} \to \mathbf{R}$ be constant and $M \subset \mathbf{R}$ be $M = (-2,-1) \cup (1,2)$.

(b) STATEMENT: *Every metric space is homeomorphic to one of finite diameter.*
TRUE. Any metric space (M, d) is homeomorphic to the same space $(M, \tilde{d})$ with bounded metric $\tilde{d}$. For example, putting

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

works because the identity map $\iota : (M, d) \to (M, \tilde{d})$ and it inverse ι^{-1} are continuous.

(c) STATEMENT: *The set* $A = \{(x_1, x_2, \ldots) \in \ell_2 : |x_i| \le 1$ *for all* $i.\}$ *is compact in* ℓ_2.
FALSE. The set contains the sequence (e_n) where $e_n = (0, \ldots, 0, 1, 0, \ldots)$ wth the "1" in the n-th slot. Now $\|e_j - e_k\|_2 = \sqrt{2}$ if $j \ne k$, so that any ball with radius $r < \sqrt{2}/2$ can contain at most one of the members of the sequence. In particular, the sequence, and hence A cannot be covered by finitely many balls of radius $r < \sqrt{2}/2$. A is not totally bounded so not compact.

4. *For* $x \in \mathbf{R}$, *let* $T : \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ *be defined by* $T[y](t) = t + \int_0^t s\, f(s)\, ds$.

Show that T *satisfies the hypotheses of the Contraction Mapping Theorem. Show that the fixed point is a solution to the differential equation* $f'(t) = t f(t) + 1$.

Let $X = \mathcal{C}([0,1])$ and $\| \bullet \|$ denote the sup-norm on $[0,1]$ making $(X, \| \bullet \|)$ a complete normed linear space (Banach Space). To show that $T : X \to X$, for $y \in X$ we have $t\, y(t)$ is a continuous function on $[0,1]$ so its indefinite integral is continuous. Adding the continuous function t yields the transformation

$$T[y](t) = t + \int_0^t s\, f(s)\, ds$$

which maps y to a continuous function $T[y]$. To show that T satisfies the contraction mapping hypotheses, we must also show that T is a contraction on X. Fix $t \in [0,1]$ and pick $x, y \in X$. Estimating

$$\begin{aligned}
\big|T[x](t) - T[y](t)\big| &= \left| t + \int_0^t s\, x(s)\, ds - t - \int_0^t s\, y(s)\, ds \right| \\
&= \left| \int_0^t s\, \big(x(s) - y(s)\big)\, ds \right| \\
&\le \int_0^t s\, |x(s) - y(s)|\, ds \\
&\le \int_0^t s\, \|x - y\|\, ds \\
&= \frac{t^2}{2} \|x - y\| \\
&\le \frac{1}{2} \|x - y\|.
\end{aligned}$$

Taking supremum on the left side over $t \in [0,1]$ yields

$$\|T[x] - T[y]\| \le \frac{1}{2} \|x - y\|,$$

showing that T is a contraction with constant $\frac{1}{2}$.

By the Contraction Mapping Theorem, there is a unique $z \in X$ such that $z = T[z]$. In other words

$$z(t) = t + \int_0^t s\, z(s)\, ds.$$

Since $t\,z(t)$ is continuous, by the Fundamental Theorem of Calculus, the indefinite integral is continuously differentiable, as is t, so that their sum may be differentiated to yield the solution of the desired ODE

$$\frac{dz}{dt}(t) = 1 + t\,z(t).$$

Evaluating at zero, we also have the initial value $z(0) = 0$.

5. *Show ℓ_∞ is complete.*

Recall that $\ell_\infty = \{f : \mathbb{N} \to \mathbf{R} : \|f\| < \infty\}$ is the set of bounded sequences of reals, where the norm is the sup norm $\|f\| = \sup\{|f(k)| : k \in \mathbb{N}\}$. To show ℓ_∞ is complete it has to be shown that every Cauchy Sequence (x_n) in ℓ_∞ converges in ℓ_∞. Choose a Cauchy Sequence (x_n) in ℓ_∞.

First we construct a candidate for the limit. Fix any $k \in \mathbb{N}$. We observe that since (x_n) is Cauchy in ℓ_∞, for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \varepsilon \qquad \text{whenever } m, n \geq N.$$

Hence

$$|x_n(k) - x_m(k)| \leq \|x_n - x_m\| < \varepsilon \qquad \text{whenever } m, n \geq N.$$

Thus the real sequence $(x_n(k))$ is a Cauchy Sequence. Since the reals are complete, there is a real limit

$$x(k) = \lim_{n\to\infty} x_n(k).$$

x is our candidate for the limit of (x_n).

Second, we show that $x \in \ell_\infty$. We know that a Cauchy sequence is bounded. Thus there is $M < \infty$ such that $\|x_n\| \leq M$ for all n. Thus for each k, $|x_n(k)| \leq \|x_n\| \leq M$. Passing the inequality to the limit, for each k,

$$|x(k)| = \lim_{n\to\infty} |x_n(k)| \leq M$$

so x is a bounded sequence or $x \in \ell_\infty$. Taking sup over k implies $\|x\| \leq M$.

Third, we must show that $x_n \to x$ in ℓ_∞. Use the fact that (x_n) is a Cauchy sequence. We have for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \varepsilon \qquad \text{whenever } m, n \geq N.$$

For any $k \in \mathbb{N}$,

$$|x_n(k) - x_m(k)| \leq \|x_n - x_m\| < \varepsilon \qquad \text{whenever } m, n \geq N.$$

Taking the limit $n \to \infty$ implies

$$|x(k) - x_m(k)| = \lim_{n\to\infty} |x_n(k) - x_m(k)| \leq \varepsilon \qquad \text{whenever } m \geq N.$$

Taking sup over k implies

$$\|x - x_m\| \leq \varepsilon \qquad \text{whenever } m \geq N,$$

which is the statement that $x_m \to x$ in ℓ_∞.