Math 5210 § 2. **Third Midterm Exam** Name: Solutions

Treibergs Apr. 2, 2025

1. *Let the partial sum* $S_n(x) = \sum_{k=1}^{n} \frac{(\sin kx)}{2^k}$. *Show that the limit exists:* $L = \lim_{n\to\infty} S_n\left(\frac{\pi n}{2n+1}\right)$.

   Note that
$$\left|\frac{(\sin kx)}{2^k}\right| \le \frac{1}{2^k}$$
   and
$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1 < \infty.$$
   So, by the Weierstrass M-test, the sum converges uniformly
$$S_n(x) \rightrightarrows S(x) = \sum_{k=1}^{\infty} \frac{(\sin kx)}{2^k}.$$
   As a uniform limit of continuous $S_n(x)$, we have $S(x)$ is continuous.

   To see that the limit exists, choose $\varepsilon > 0$. By the continuity of $S(x)$ at $\pi/2$, there is a $\delta > 0$ so that
$$\left|S(x) - S(\frac{\pi}{2})\right| < \frac{\varepsilon}{2} \qquad \text{whenever } |x - \frac{\pi}{2}| < \delta.$$
   Let
$$x_n = \frac{\pi n}{2n+1}.$$
   Since $x_n \to \frac{\pi}{2}$ as $n \to \infty$, there is N_1 such that $\left|x_n - \frac{\pi}{2}\right| < \delta$ whenever $n \ge N_1$.

   By uniform convergence, there is an $N_2 \in \mathbf{R}$ such that
$$|S_m(x) - S(x)| < \frac{\epsilon}{2} \qquad \text{whenever } x \in \mathbf{R} \text{ and } n \ge N_2.$$
   Let $N_3 = \max\{N_1, N_2\}$. For any $n \ge N_3$,
$$\left|S_n(x_n) - S(\frac{\pi}{2})\right| \le |S_n(x_n) - S(x_n)| + \left|S(x_n) - S(\frac{\pi}{2})\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$
   This proves that
$$L = \lim_{n\to\infty} S_n\left(\frac{\pi n}{2n+1}\right) = S\left(\frac{\pi}{2}\right) = \sum_{j=1}^{\infty} \frac{(-1)^j}{2^{2j-1}} = \frac{2}{5}.$$

2. *State the Arzela-Ascoli Theorem for subsets* $\mathcal{S} \subset \mathcal{C}[0,1]$.

   Let $\mathcal{T} : \mathcal{C}[0,1] \to \mathcal{C}([0,1]$ *be defined by* $\mathcal{T}[f](x) = \int_0^1 \sin(x-t)\, f(t)\, dt$.

   For $M \in \mathbf{R}$, *let* $\mathcal{B} = \{f \in \mathcal{C}[0,1] : \|f\|_\infty \le M\}$. *Show that every sequence* $(g_n) \subset \mathcal{T}[\mathcal{B}]$ *has a convergent subsequence.*

   Arzela-Ascoli Theorem. Let $\mathcal{S} \subset \mathcal{C}[X]$ where X is a compact metric space. Then $\mathcal{S}$ is compact if and only if it is closed, uniformly bounded and equicontinuous.

For us, $X = [0,1]$. Choose $f_n \in \mathcal{B}$ such that $g_n = \mathcal{T}[f_n]$. Then for any x

$$\begin{aligned} |g_n(x)| =& |\mathcal{T}[f_n](x)| = \left| \int_0^1 \sin(x-t)\, f_n(t)\, dt \right| \\ & \le \int_0^1 |\sin(x-t)|\, |f_n(t)|\, dt \le \int_0^1 \|f_n\|\, dt \le \int_0^1 M\, dt = M. \end{aligned}$$

Since this applies to all n and x, we see that (g_n) is uniformly bounded.

For any $x, y \in [0,1]$

$$\begin{aligned} |g_n(x) - g_n(y)| =& |\mathcal{T}[f_n](x) - \mathcal{T}[f_n](y)| = \left| \int_0^1 \sin(x-t)\, f_n(t)\, dt - \int_0^1 \sin(y-t)\, f_n(t)\, dt \right| \\ & \le \int_0^1 |\sin(x-t) - \sin(y-t)|\, |f_n(t)|\, dt \le \int_0^1 |x-y|\, \|f_n\|\, dt \\ & \le \int_0^1 |x-y|\, M\, dt = M|x-y|, \end{aligned}$$

where we have used $|\sin(x-t) - \sin(y-t)| \le |x-y|$. Since this applies to all n, x and y, we see that (g_n) are uniformly M-Lipschitz, thus uniformly equicontinuous.

By the Arzela-Ascoli Theorem, unifolmly bounded and equicontinuous sequences (g_n) have a convergent subsequence in $\mathcal{C}[0,1]$.

It turns out that $\mathcal{T}[\mathcal{B}]$ is not closed.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT: *The nth Bernsten operator* $\mathcal{B}_n[f](x) = \sum_{k=0}^n f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$ *is a bounded operator* $\mathcal{B}_n : \mathcal{C}[0,1] \to \mathcal{C}([0,1]$.

   TRUE. For $f \in \mathcal{C}[0,1]$ and any x we have

$$\begin{aligned} |\mathcal{B}_n[f](x)| &= \left| \sum_{k=0}^n f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \right| \\ &\le \sum_{k=0}^n \left| f\left(\frac{k}{n}\right) \right| \binom{n}{k} x^k (1-x)^{n-k} \\ &\le \sum_{k=0}^n \|f\| \binom{n}{k} x^k (1-x)^{n-k} = \|f\|. \end{aligned}$$

   Taking supremum over x, $\|\mathcal{B}_n[f]\| \le \|f\|$. This is true for all f so the operator is bounded and its operator norm $\|\mathcal{B}_n\| \le 1$. In fact the operator norm equals one because $\mathcal{B}_n[\text{const.}](x) = \text{const.}$

   (b) STATEMENT: $\|\bullet\|_1$ *and* $\|\bullet\|_2$ *are equivalent norms on* $\mathbf{R}^n$.

   TRUE. There are positive constants c_1, c_2 such that $c_1\|\bullet\|_1 \le \|\bullet\|_2 \le c_2\|\bullet\|_1$. To see this, from the definitions of norms for $x \in \mathbf{R}^n$ and the Schwartz Inequality,

$$\|x\|_1 = \sum_{k=1}^n |x_k| = \sum_{k=1}^n 1 \cdot |x_k| \le \|(1,1,\ldots,1)\|_2\, \|x\|_2 = \sqrt{n}\|x\|_2;$$

$$\|x\|_2^2 = \sum_{k=1}^n |x_k|^2 \le \left(\sum_{k=1}^n |x_k| \right)^2 = \|x\|_1^2.$$

   Thus the norms are equivalent with $c_1 = 1/\sqrt{n}$ and $c_2 = 1$.

(c) STATEMENT: *F_σ sets are closed.*

FALSE. Countable unions of closed sets need not be closed, for example

$$\bigcup_{n=1}^{\infty}\left[0, \frac{n}{n+1}\right] = [0,1).$$

4. *State the Stone Weierstrass Theorem for real functions of $\mathcal{C}[0,1]$. Let*

$$\mathcal{H} = \left\{ \sum_{k=0}^{n} c_k \, \cos(kx) : c_k \in \mathbf{R} \right\}$$

be the set of cosine polynomials. Show that $\mathcal{H}$ is an algebra. Is $\mathcal{H}$ dense in $\mathcal{C}[0,1]$? Is $\mathcal{H}$ dense in $\mathcal{C}[-1,1]$?

Stone Weierstrass Theorem for Real Scalars. Let $\mathcal{A} \subset \mathcal{C}[X]$ be a subalgebra of the real valued continuous functions on a compact metric space X. If $\mathcal{A}$ separates points in X and vanishes at no point of X, then $\mathcal{A}$ is dense in $\mathcal{C}[X]$.

For us $X = [0,1]$ or $[-1,1]$. To show that $\mathcal{H}$ is an algebra, we must first show that it is a vector subspace. It suffices to show that H is closed under linear combinations. If $f, g \in \mathcal{H}$ and $\alpha, \beta \in \mathbf{R}$, then

$$f(x) = \sum_{k=0}^{n} c_k \, \cos(kx),$$
$$g(x) = \sum_{k=0}^{m} d_k \, \cos(kx),$$

where c_k and d_k are real. For convenience, put $c_k = 0$ for $k > n$, $d_k = 0$ for $k > m$ and $\ell = \max\{n, m\}$. Then

$$\alpha f + \beta g = \alpha \left(\sum_{k=0}^{n} c_k \, \cos(kx) \right) + \beta \left(\sum_{k=0}^{m} d_k \, \cos(kx) \right) = \sum_{k=0}^{\ell} (\alpha c_k + \beta d_k) \cos kx$$

is a cosine polynomial. Second, we must show products of cosine polynomials are cosine polynomials.

$$fg = \left(\sum_{k=0}^{n} c_k \, \cos(kx) \right) \left(\sum_{p=0}^{m} d_p \, \cos(px) \right) = \sum_{k=0}^{n} \sum_{p=0}^{m} c_k \, d_p \, \cos(kx) \, \cos(px)$$

We can express the product of cosines as a sum using a trig identitiy.

$$\cos(k+p)x = \cos kx \cos px - \sin kx \sin px$$
$$\cos(p-k)x = \cos kx \cos px + \sin kx \sin px$$

thus, adding,

$$\cos kx \cos px = \frac{1}{2}\big(\cos(k+p)x + \cos(p-k)x\big).$$

Thus, the cosine polynomials are closed under linear combinations and products, *i.e.,* $\mathcal{H}$ is a subalgebra.

$1 \in \mathcal{H}$ so $\mathcal{H}$ does not vanish at a point. If $X = [0,1]$ then $\cos x$ is strictly decreasing, so for any $x \neq y$ in $[0,1]$, $\cos x \neq \cos y$ so it separates points. By the Stone Weierstrass Theorem, $\mathcal{H}$ is dense in $\mathcal{C}[0,1]$.

The functions of $\mathcal{H}$ are even, so they don't separate points of $\mathcal{C}[-1,1]$ because $f(-1) = f(1)$ for every $f \in \mathcal{H}$. $\mathcal{H}$ is not dense in $\mathcal{C}[-1,1]$. Indeed x can't be approximated because $\|x - f\|_\infty \ge \max\{|-1 - f(-1)|, |1 - f(1)|\} = \max\{|-1 - f(1)|, |f(1) - 1|\}$
$\ge \frac{1}{2}(|-1 - f(1)| + |f(1) - 1|) \ge \frac{1}{2}|-1-1| = 1$ for all $f \in \mathcal{H}$.

5. *Let* $E \subset \mathbf{R}$. *Define the* Lebesgue outer measure $\mathrm{m}^*(E)$. *Let* $E \subset \mathbf{R}$. *Define what it means for* E *to be* Lebesgue measurable. *Let* $E \subset \mathbf{R}$ *be a null set* $\mathrm{m}^*(E) = 0$ *and* $S \subset E$. *Show that* S *is Lebesgue measurable. (You may assume* m^* *is countably subadditive.)*

For a set $E \in \mathbf{R}$, the *Lebesgue Outer Measure* is

$$\mathrm{m}^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) : E \subset \bigcup_{k=1}^{\infty} I_k \right\}$$

where the infimum is taken over all coverings of E by countable unions of intervals and $\ell(I_k)$ denotes the length of the interval I_k.

A set $E \subset \mathbf{R}$ is *Lebesgue Measurable* if for each $\varepsilon > 0$ there is a closed set F and an open G with $F \subset E \subset G$ such that $\mathrm{m}^*(G \backslash F) < \varepsilon$.

To show that S is measurable, lets take $F = \emptyset$. Since $\mathrm{m}^*(E) = 0$, by the definition of outer measure, there are intervals I_k with endpoints $\alpha_k \le \beta_k$ such that $E \subset \bigcup_{k=1}^\infty I_k$ with $\sum_{k=1}^\infty \ell(I_k) < \frac{\varepsilon}{2}$. Let J_k be a slightly larger open interval

$$J_k = \left(\alpha_k - \frac{\varepsilon}{2^{k+2}}, \beta_k + \frac{\varepsilon}{2^{k+2}} \right).$$

Then $I_k \subset J_k$ and $\ell(J_k) = \ell(I_k) + \frac{\varepsilon}{2^{k+1}}$. Then the union $G = \bigcup_{k=1}^\infty J_k$ is an open set such that $F = \emptyset \subset S \subset E \subset G$. By the definition of outer measure

$$\begin{aligned} \mathrm{m}^*(G \backslash F) = \mathrm{m}^*(G) = \mathrm{m}^* \left(\bigcup_{k=1}^{\infty} J_k \right) &\le \sum_{k=1}^{\infty} \ell\left(J_k\right) \\ &\le \sum_{k=1}^{\infty} \left[\ell(I_k) + \frac{\varepsilon}{2^{k+1}} \right] = \left(\sum_{k=1}^{\infty} \ell(I_k) \right) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus S is maesurable.