Math 5410 § 1. **First Midterm Exam** Name: Solutions
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1. *For the differential equation, find the general solution. Find the Poincaré map. Show that there is a unique 2π-periodic solution.*

$$x' = (1 + \cos t)x + 1.$$

Use the integrating factor $\mu(t) = \exp\left(-\int_0^t 1 + \cos s\, ds\right) = e^{-t-\sin t}$.

$$\left(e^{-t-\sin t}x\right)' = e^{t+\sin t}\left(x' - (1+\cos t)x\right) = e^{-t-\sin t}$$

Integrating,

$$e^{-t-\sin t}x(t) - e^0 x(0) = \int_0^t e^{-s-\sin s}\, ds$$

so the general solution has the form

$$x(t) = e^{t+\sin t}x(0) + e^{t+\sin t}\int_0^t e^{-s-\sin s}\, ds.$$

The Poincaré map is the value that the solution starting from x_0 has at the time of one period $T = 2\pi$. Hence

$$\wp(x_0) = e^{2\pi+\sin 2\pi}x_0 + e^{2\pi+\sin 2\pi}\int_0^{2\pi} e^{-s-\sin s}\, ds = e^{2\pi}x_0 + e^{2\pi}\int_0^{2\pi} e^{-s-\sin s}\, ds.$$

There is a unique 2π-periodic solution if there is exactly one starting value that returns to itself after one period, namely, a there is exactly one solution to the equation $\wp(x_0) = x_0$. This is a linear equation in x_0 so it has the unique solution

$$x_0 = \frac{e^{2\pi}}{1 - e^{2\pi}}\int_0^{2\pi} e^{-s-\sin s}\, ds.$$

2. *Suppose that a population grows according to the logistic model but is harvested at a rate proportional to the population, where $h > 0$ is the harvesting parameter. Find the bifurcation points and sketch the phase lines for values of h just above and just below the bifurcation values. Sketch the bifurcation diagram for this family of differential equations. Is the initial population exterminated or does it have a positive limiting value in these cases?*

$$x' = x(1-x) - hx.$$

The fixed points are solutions of $0 = x(1-h-x)$ which are $x = 0$ and $x = 1-h$. There is one bifurcation at $h = 1$ which is of trans-critical type. In the $x - h$ plane, the bifurcation diagram consits of the lines $x = 0$ and $x = 1-h$. For $h > 1$, the one zero is negative and unphysical. Between the roots the right side is positive and the flow is increasing. At the bifurcation point $h = 1$, the flow is decreasing on both sides and the rest point is neither stable nor unstable. For $h < 1$ the rest points are zero and $x = 1-h$ which is positive. The flow is increasing to the left and decreasing to the right. The stable fixed points are red and the unstable ones are blue in the diagram. The phase lines corresponding to the three values $h > 1$, $h = 1$ and $h < 1$ are drawn showing the flows.

When the harvesting rate is small $h < 1$, then there is a stable positive equilibrium at $x = 1-h$. Otherwise the population decays to zero. Because zero is a rest point, the extermination takes infinite time.

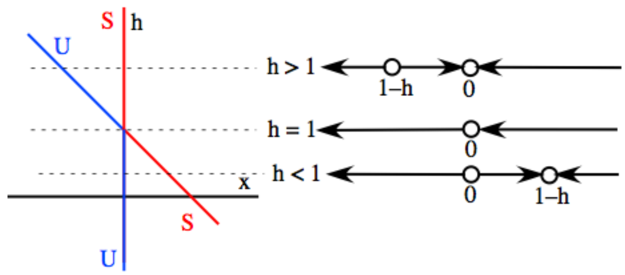


Figure 1: Bifurcation Diagram and Phase Lines for Problem (2).

3. *In the system, for which real values of k does the system have complex eigenvalues? real and distinct eigenvalues? Identify regions of the k-axis where the system has similar phase portraits. In each of the regions, what is the canonical form? For each, sketch the phase plane showing the trajectories. Identify the regions in the k-axis where the system has TOPOLOGICALLY CONJUGATE phase portraits.*

$$X' = \begin{pmatrix} 1 & k \\ 2 & 3 \end{pmatrix} X$$

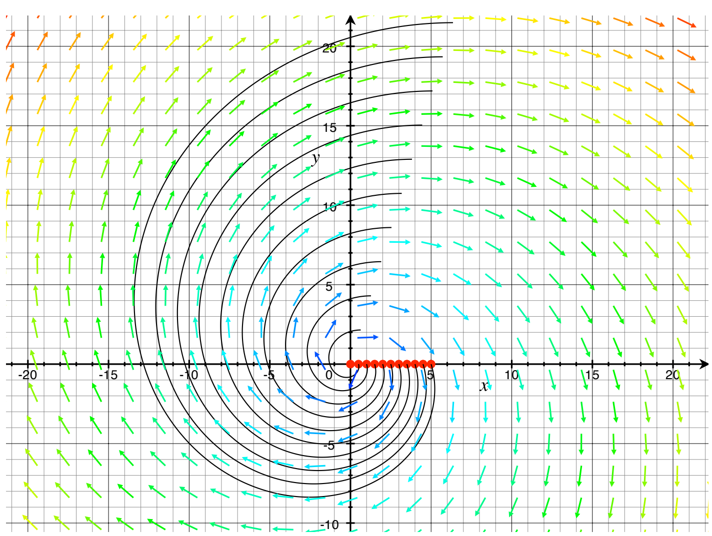
The characteristic polynomial is

$$0 = \det(A - \lambda I) = \left| \begin{pmatrix} 1-\lambda & k \\ 2 & 3-\lambda \end{pmatrix} \right| = (1-\lambda)(3-\lambda) - 2k = \lambda^2 - 4\lambda + 3 - 2k.$$

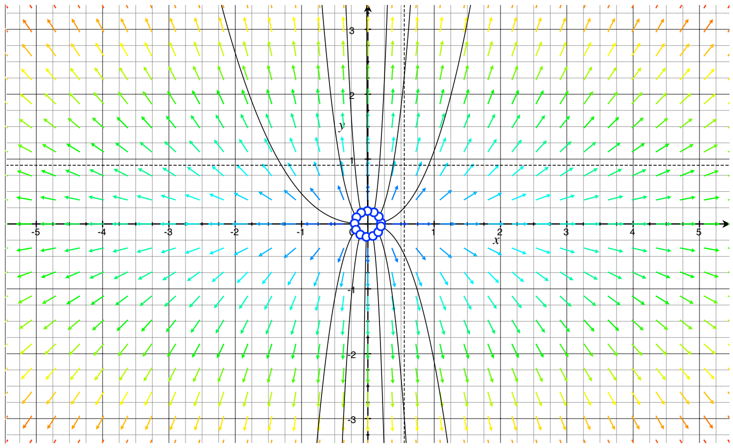
Its zeros are

$$\lambda = \frac{4 \pm \sqrt{16 - 4(3-2k)}}{2} = 2 \pm \sqrt{1+2k}.$$

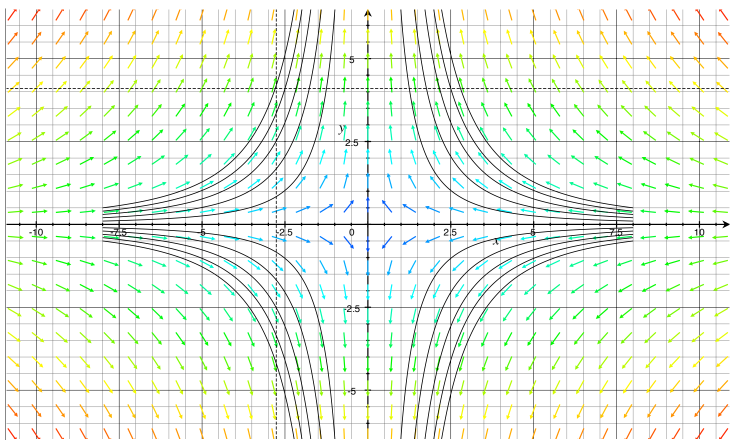
Thus the eigenvalues are complex if $1+2k < 0$ or $k < -\frac{1}{2}$ and the real part is 2. If $1+2k > 0$ there are distinct real eigenvalues. For $0 < 1+2k < 4$ the eigenvalues are both positive. If $1+2k > 4$ then the square root exceeds 2 and the eigenvalues have opposite signs. Hence, in the k-axis, the solutions are unstable spirals if $k < -\frac{1}{2}$. The solutions form an unstable (improper) node if $-\frac{1}{2} < k < \frac{3}{2}$. The solutions are saddles if $\frac{3}{2} < k$.



Complex eigenvalues if $k < -\frac{1}{2}$.



Real positive distinct eigenvalues if $-\frac{1}{2} < k < \frac{3}{2}$.



One negative and one positive eigenvalue if $\frac{3}{2} < k$.

The canonical form in case $k < -\frac{1}{2}$ is the complex form with $a = 2$ and $b = \sqrt{-1-2k}$. In case $-\frac{1}{2} < k$ we let $\lambda_1, \lambda_2 = 1 \pm \sqrt{1+2k}$. If $-\frac{1}{2} < k < \frac{3}{2}$ then $0 < \lambda_1 < \lambda_2$ and the phase portrait is an unstable (improper) node. If $k > \frac{3}{2}$ then $\lambda_1 < 0 < \lambda_2$ and the phase portrait is a saddle. The canonical forms are, resp.,

$$Y' = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} Y; \qquad Z' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Z; \qquad W' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} W.$$

The canonical forms with both eigenvalues with positive real parts are topologically conjugate. Thus if $k < \frac{3}{2}$ then all phase portraits are topologically conjugate unstable points. Also for $\frac{3}{2} < k$ the phase portraits are all saddles, thus topologically conjugate to each other, but not to the unstable points.

4. *Consider the scalar equation $x'' - 4x' + 4x = 0$. Convert to an equivalent system $X' = AX$. Find the eigenvalues and eigenvectors of A. Find a matrix T so that the change of variables $X = TY$ puts the system into canonical form $Y' = BY$. Check that your matrix does the job by using your T to make the change of variables.*

   By letting $y = x'$ we get the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ 4y - 4x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

   The eigenvalues are gotten by solving

$$0 = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -4 & 4-\lambda \end{vmatrix} = (-\lambda)(4-\lambda) + 4 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$$

   thus we have a double eigenvalue $\lambda = 2, 2$. The eigenvector V for $\lambda = 2$ is found by

$$0 = (A - \lambda I)V = \begin{pmatrix} -2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

   Using the recipe in the text, we choose any vector independent of V, say $W = \binom{1}{0}$ and compute

$$\mu V + \nu W = AW = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

   Then the matrix T consists of columns V and $\frac{1}{\mu}W$, namely

$$T = (V; \frac{1}{\mu}W) = \begin{pmatrix} 1 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix}.$$

   The change of variable $X = TY$ means

$$TY' = X' = AX = ATY \qquad \Longrightarrow \qquad Y' = T^{-1}ATY.$$

   To see that it does the job,

$$T^{-1}AT = \begin{pmatrix} 0 & -\frac{1}{2} \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

   There is another recipe involving a cyclic vector U, namely a soluton

$$(A - \lambda I)U = V \qquad \Longrightarrow \qquad \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

   Then the matrix T consists of columns V and U. In this case

$$T^{-1}AT = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

5. *Show that the function*

$$\varphi^A\big(t, (\alpha, \beta)\big) = e^{-t} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

   is the flow induced by the system

$$X' = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} X.$$

Find $\tau(\alpha,\beta)$, the time at which the flow φ^A starting at the point $(\alpha,\beta)\in\mathbf{R}^2$ meets the unit circle $\mathbb{S}^1$. Find the flow $\varphi^B\big(t,(\gamma,\delta)\big)$ of the system

$$Y' = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} Y.$$

What is a homeomorphism $h : \mathbf{R}^2 \to \mathbf{R}^2$ that shows the the systems are topologically conjugate? What equation must h satisfy? (Simplify the formulas as much as possible but you do not need to check that your equation holds.)

To see that φ^A is the flow of the system, we need to check that $\varphi^A(0,(\alpha,\beta)) = (\alpha,\beta)$ and that $\frac{d}{dt}\varphi^A = A\varphi^A$. First

$$\varphi^A\big(0,(\alpha,\beta)\big) = e^{-0}\begin{pmatrix} \cos 0 & \sin 0 \\ -\sin 0 & \cos 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Second

$$\begin{aligned}
\frac{d}{dt}\varphi^A\big(t,(\alpha,\beta)\big) &= \frac{d}{dt}e^{-t}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= e^{-t}\begin{pmatrix} -\cos t - \sin t & -\sin t + \cos t \\ \sin t - \cos t & -\cos t - \sin t \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} e^{-t}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= A\varphi^A\big(t,(\alpha,\beta)\big).
\end{aligned}$$

The system $Y' = BY$ decouples into $x' = -x$ and $y' = -y$ so the flow is given by

$$\varphi^B\big(t,(\alpha,\beta)\big) = e^{-t}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

The time $\tau(\alpha,\beta)$ to reach the unit circle is the solution of

$$\begin{aligned}
1 = |\varphi^A\big(\tau,(\alpha,\beta)\big)| &= \left| e^{-\tau}\begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right| \\
&= e^{\tau}\sqrt{(\alpha\cos\tau + \beta\sin\tau)^2 + (-\alpha\sin\tau + \beta\cos\tau)^2} \\
&= e^{-\tau}\sqrt{\alpha^2+\beta^2}.
\end{aligned}$$

Hence $\tau = \log r$ where $r(\alpha,\beta) = \sqrt{\alpha^2+\beta^2}$.

Then the homeomorphism is defined by the recipe in the text

$$\begin{aligned}
h(\alpha,\beta) &= \varphi^B_{-\tau}\circ\varphi^A_{\tau}(\alpha,\beta) \\
&= e^{\tau}\cdot e^{-\tau}\begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}
\end{aligned}$$

where $\tau = \tau(\alpha,\beta)$ for short.

It is a homeomorphism defined on all of $\mathbf{R}^2$. It must satisfy for all $t,\alpha,\beta\in\mathbf{R}$,

$$h\circ\varphi^A_t(\alpha,\beta) = \varphi^B_t\circ h(\alpha,\beta). \tag{1}$$

This completes the solutions of the exam. As an extra, we provide the check that indeed equation (1) holds. First we recall that

$$\tau\big(\varphi_t^A(\alpha,\beta)\big) = \tau(\alpha,\beta) - t.$$

This is because of the semigroup property

$$\varphi^A_{\tau(\alpha,\beta)}(\alpha,\beta) = \varphi^A_{\tau(\varphi_t^A(\alpha,\beta))} \circ \varphi_t^A(\alpha,\beta) = \varphi^A_{\tau(\alpha,\beta)-t} \circ \varphi_t^A(\alpha,\beta).$$

It can also be seen by

$$r\big(\varphi_t^A(\alpha,\beta)\big) = \left| e^{-t}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right| = e^{-t}\sqrt{\alpha^2+\beta^2}$$

so that

$$\tau\big(\varphi_t^A(\alpha,\beta)\big) = \log(e^{-t}\sqrt{\alpha^2+\beta^2}) = \log(\sqrt{\alpha^2+\beta^2}) - t = \tau(\alpha,\beta) - t.$$

Now we check (1). Using $\tau' = \tau\big(\varphi_t^A(\alpha,\beta)\big) = \tau(\alpha,\beta) - t$ and $\tau = \tau(\alpha,\beta)$ so $\tau' + t = \tau$,

$$\begin{aligned}
h \circ \varphi_t^A(\alpha,\beta) &= \begin{pmatrix} \cos\tau' & \sin\tau' \\ -\sin\tau' & \cos\tau' \end{pmatrix} e^{-t}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= e^{-t}\begin{pmatrix} \cos\tau'\,\cos t - \sin\tau'\,\sin t & \cos\tau'\,\sin t + \sin\tau'\,\cos t \\ -\sin\tau'\,\cos t - \cos\tau'\,\sin t & -\sin\tau'\,\sin t + \cos\tau'\,\cos t \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= e^{-t}\begin{pmatrix} \cos(\tau'+t) & \sin(\tau'+t) \\ -\sin(\tau'+t) & \cos(\tau'+t) \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= e^{-t}\begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= \varphi_t^B \circ h(\alpha,\beta).
\end{aligned}$$