Math 5410 § 1. **First Midterm Exam** Name: Practice Problems
Treibergs September 19, 2014

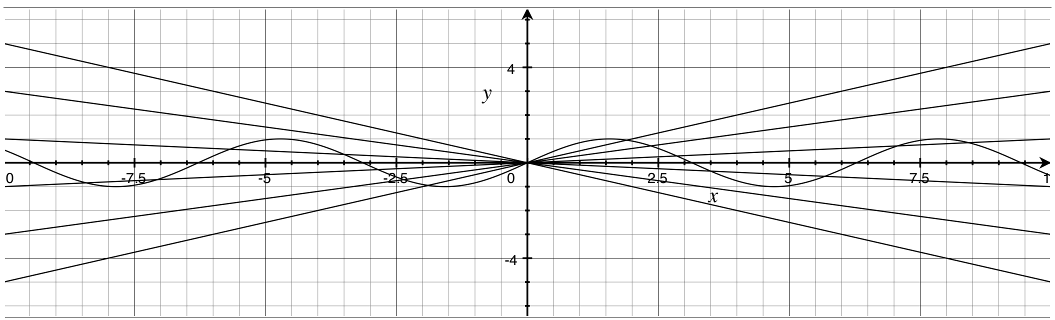
1. *Consider the family of differential equations for the parameter a:*

$$x' = ax + \sin x.$$

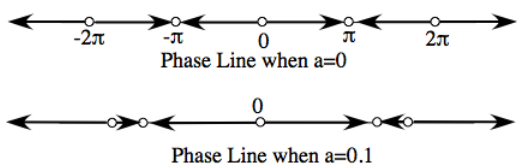
   (a) Sketch the phase line when $a = 0$.

   (b) Use the graphs of ax and $\sin x$ to determine the qualitative behavior of all bifurcations that occur as a increases form -1 to 1.

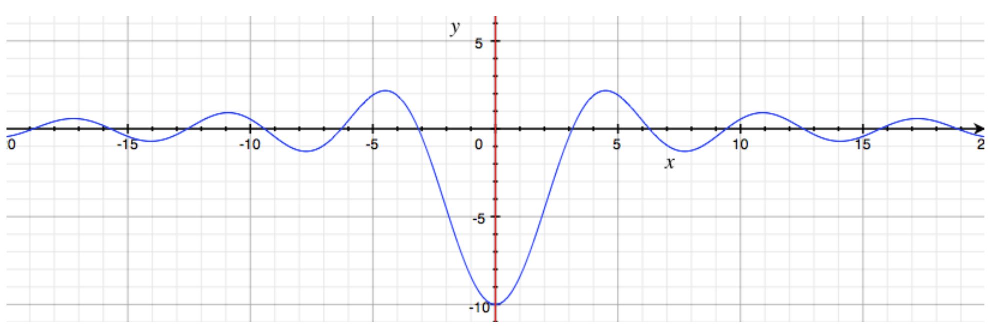
   (c) Sketch the bifurcation diagram for this family of differential equations.



The equations $y = \sin x$ and $y = ax$ for $a = \pm.1, \pm.3, \pm.5$ are superimposed. The zeros of $ax + \sin x$ are the intersection points. So when $a = 0$ the rest points are at πk for integer k and the flow directions alternate in each interval. As a moves from zero, the line $y = -ax$ intersects $y = \sin x$ at finitely many and fewer and fewer points. When $a = .1$ then there are only five rest points. The stable/unstable pairs move toward each other as a increases and vanish.



The bifurcation diagram are the solutions of $a + \frac{\sin x}{x} = 0$, which are plotted as the blue and red curves. It shows how as a departs from $a = 0$ and moves to $|a| = 1$, there are fewer and fewer rest points that such that sources and sinks cancel pairwise as $|a|$ increases. For $a > -1$ near $a = -1$ there are only three rest points which collapse to one in a pitchfork bifurcation at $x = 0$ and $a = -1$. After $|a| \geq 1$ there is only one rest point at 0.



2. *Solve the initial value problem:*

$$X' = \begin{pmatrix} -5 & 3 \\ 9 & 1 \end{pmatrix} X; \qquad X(0) = \begin{pmatrix} 5 \\ 6 \end{pmatrix}.$$

The characteristic equation is

$$0 = \det(A - \lambda I) = (-5 - \lambda)(1 - \lambda) - 3 \cdot 9 = \lambda^2 + 4\lambda - 32 = (\lambda - 4)(\lambda + 8).$$

Hence the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -8$. The eigenvactors is

$$0 = (A - \lambda_1 I)V_1 = \begin{pmatrix} -9 & 3 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \qquad 0 = (A - \lambda_2 I)V_2 = \begin{pmatrix} 3 & 3 \\ 9 & 9 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Then the general solution is

$$X(t) = c_1 e^{4t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{-8t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Then the initial value problem is solved by

$$X(0) = \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \implies c_1 = \frac{11}{4},\ c_2 = \frac{9}{4}.$$

3. Find the general solution:

$$X' = \begin{pmatrix} -4 & -1 \\ 2 & -2 \end{pmatrix} X.$$

The characteristic equation is

$$0 = \det(A - \lambda I) = (-4 - \lambda)(-2 - \lambda) - (-1) \cdot 2 = \lambda^2 + 6\lambda + 10.$$

Hence the eigenvalues are $\lambda = -3 \pm i$. An eigenvactor for $\lambda = -3 + i$ is

$$0 = (A - \lambda I)V = \begin{pmatrix} -1 - i & -1 \\ 2 & 1 - i \end{pmatrix}\begin{pmatrix} -1 \\ 1 + i \end{pmatrix}$$

A complex solution is

$$\begin{aligned} X(t) &= e^{(-3+i)t}\begin{pmatrix} -1 \\ 1 + i \end{pmatrix} = e^{-3t}(\cos t + i \sin t)\begin{pmatrix} -1 \\ 1 + i \end{pmatrix} \\ &= e^{-3t}\begin{pmatrix} -\cos t \\ \cos t - \sin t \end{pmatrix} + ie^{-3t}\begin{pmatrix} -\sin t \\ \cos t + \sin t \end{pmatrix} \end{aligned}$$

The real and imaginary parts are independent solutions so the general solution is

$$X(t) = c_1 e^{-3t}\begin{pmatrix} -\cos t \\ \cos t - \sin t \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} -\sin t \\ \cos t + \sin t \end{pmatrix}.$$

4. Find a matrix T that brings the equation into canonical form. Show that your change of variables does the job.

$$X' = \begin{pmatrix} 4 & 4 \\ -1 & 0 \end{pmatrix} X.$$

The characteristic equation is

$$0 = \det(A - \lambda I) = (4 - \lambda)(-\lambda) - (-1) \cdot 4 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.$$

Hence the eigenvalues are $\lambda = 2, 2$. An eigenvactor for $\lambda = 2$ is

$$0 = (A - \lambda I)V = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Since the matrix $A - \lambda I$ has rank one, there are no more independent eigenvectors. Then the recipe says take any other independent vector, say $W = \binom{0}{1}$, and compute

$$AW = \begin{pmatrix} 4 & 4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \mu V + \nu W = 2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then consider the matrix T whose columns are V and $(1/\mu)W$. Checking,

$$T^{-1}AT = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 4 & 4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ -1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

To see if it does the job, change variables by $X = TY$. Then

$$TY' = X' = \begin{pmatrix} 4 & 4 \\ -1 & 0 \end{pmatrix} X = AX = ATY \qquad \Longrightarrow Y' = T^{-1}ATY = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} Y$$

5. *Show that the two systems are topologically conjugate by finding a conjugating homeomorphism and checking:*

$$X' = \begin{pmatrix} 8 & 3 \\ -6 & -1 \end{pmatrix} X; \qquad Y' = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} Y.$$

Computing the eigenvalues of the first system we find the characteristic equation is

$$0 = \det(A - \lambda I) = (8 - \lambda)(-1 - \lambda) - 3 \cdot (-6) = \lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5),$$

thus the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 5$. Thus the problem asks that we show that the canonical form is topologically conjugate to the original matrix. This is achieved by a linear map that changes variables. Computing the eigenvectors we find

$$0 = (A - \lambda_1 I)V_1 = \begin{pmatrix} 6 & 3 \\ -6 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad 0 = (A - \lambda_2 I)V_2 = \begin{pmatrix} 3 & 3 \\ -6 & -6 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The general solution is thus

$$X(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The particular solution when $X(0) = \binom{\alpha}{\beta}$ is gotten by solving

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad \Longrightarrow \qquad c_1 = -\alpha - \beta, \ c_2 = 2\alpha + \beta.$$

Then the first flow may be written

$$\varphi_t^A(\alpha, \beta) = -(\alpha + \beta)e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + (2\alpha + \beta)e^{5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2e^{5t} - e^{2t} & e^{5t} - e^{2t} \\ 2e^{2t} - 2e^{5t} & 2e^{2t} - e^{5t} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

The second equation is already diagonal, so the flow may be written

$$\varphi_t^B(\gamma, \delta) = \begin{pmatrix} \gamma e^{2t} \\ \delta e^{5t} \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

Putting $T = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$ we have

$$T^{-1}AT = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 8 & 3 \\ -6 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ -4 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$$

Letting $h(X) = T^{-1}X$ we must show

$$h \circ \varphi_t^A(x_0) = \varphi_t^B \circ h(x_0)$$

where φ_t^A is the flow of the first system. Thus

$$\begin{aligned} h \circ \varphi_t^A(\alpha, \beta) &= \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2e^{5t} - e^{2t} & e^{5t} - e^{2t} \\ 2e^{2t} - 2e^{5t} & 2e^{2t} - e^{5t} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \begin{pmatrix} -e^{2t} & -e^{2t} \\ 2e^{5t} & e^{5t} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \begin{pmatrix} -(\alpha+\beta)e^{2t} \\ (2\alpha+\beta)e^{5t} \end{pmatrix} \end{aligned}$$

whereas

$$\begin{aligned} \varphi_t^B \circ h(x_0) &= \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{pmatrix}\begin{pmatrix} -\alpha - \beta \\ 2\alpha + \beta \end{pmatrix} \\ &= \begin{pmatrix} -(\alpha+\beta)e^{2t} \\ (2\alpha+\beta)e^{5t} \end{pmatrix} \end{aligned}$$

Both flows are the same!

6. *Consider the system*

$$X' = \begin{pmatrix} a & b \\ b & a \end{pmatrix} X$$

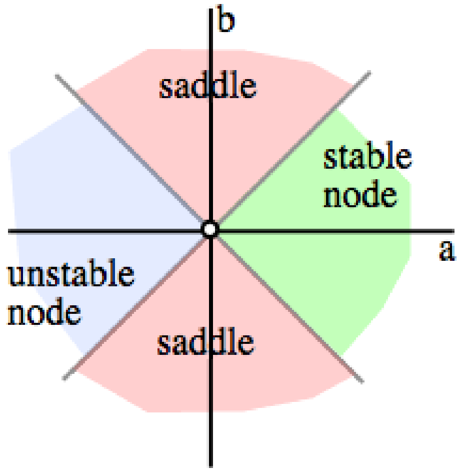
Sketch the region in the a-b plane where this system has different types of cononical forms. Find these canonical forms. Show the corresponding regions on the determinant-trace plane.

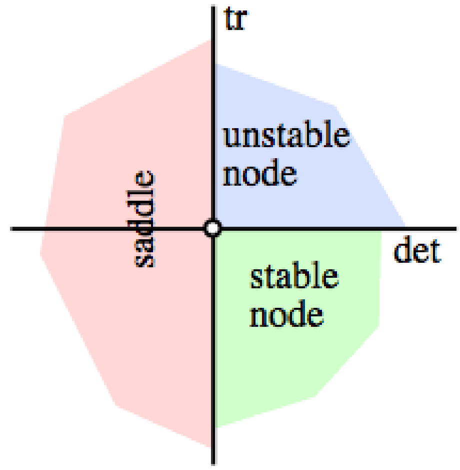
The characteristic equation is

$$0 = \det(A - \lambda I) = (a-\lambda)^2 - b^2 \qquad \Longrightarrow \qquad a - \lambda = \pm b$$

so that eigenvalues are $a \pm b$. If $|b| > |a|$ then the eigenvalues are negative and positive, and the flow is a saddle with canonical form $\begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}$. If $a > |b|$ then both eigenvalues are positive, and the flow is an unstable improper node. If $a < -|b|$ then both roots are negative and the flow is a stable improper node. Along $a = |b| > 0$ roots are zero and positive, thus flow is an "unstable brush" with canonical form $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Along $a = -|b| < 0$ the roots are zero and negative, thus flow is a "stable brush." If $b = 0$ then the node is a proper node with canonical form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. At $a = b = 0$ all points are rest points with canonical form $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

If $\det = a^2 - b^2 < 0$ then roots are opposite signe and the solution is a saddle. If $\det > 0$ then the solution is a node, unstable if $\operatorname{tr} > 0$ and stable if $\operatorname{tr} < 0$. If $\det = 0$ then one root is zero and the other has the sign of a

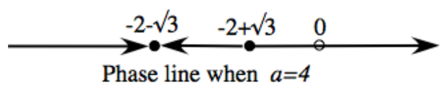




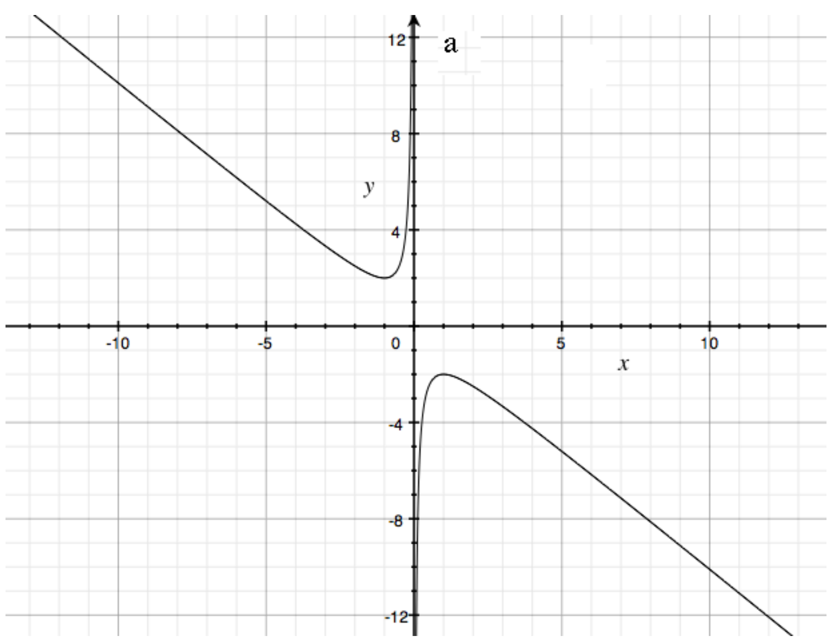
7. Sketch the phase line and the bifurcation diagram corresponding to the family of differential equations with parameter a. Find all equilibrium solutions and determine whether they are sinks, sources or neither.

$$x' = x^2 + ax + 1$$

The roots are $-\frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2-4}$. Thus bifurcation occurs at $a = \pm 2$. Stable and unstable nodes split as $|a| > 2$ increases. When $|a| > 2$ the left rest point is stable and the right is unstable. For $|a| < 2$ there are no rest points. here is a typical phase line:



Plotting the bifurcation diagram we get solving for $a = -(1+x^2)/x$. The curves locate the rest points at given a.



8. *Consider the harmonic oscillator equation with parameters $c \geq 0$ and $k > 0$*

$$x'' + cx' + kx = 0.$$

(a) *For which values of c and k does the system have complex eigenvalues? real and distinct eigenvalues? Repeated eigenvalues? identify the regions in the ck-plane where the system has similar phase phase portraits.*

(b) *In each of the cases in (a), sketch the graph showing the motion of the mass when the mass is released from an initial position with $x = 1$ and zero velocity and from an initial position with $x = 0$ and unit velocity.*

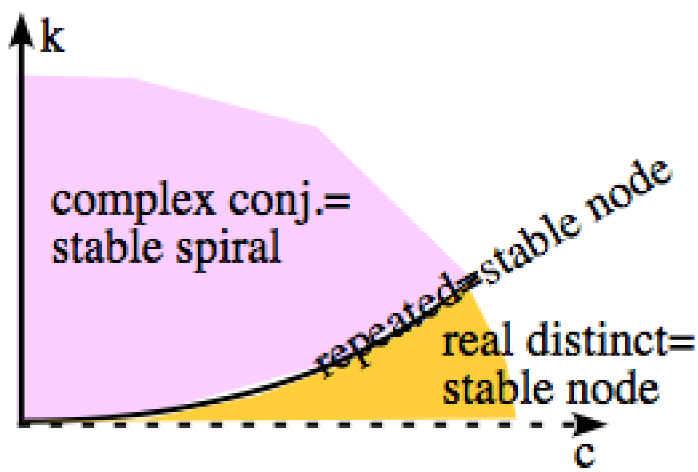
Put $x' = y$ to get system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -k & -c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The characteristic equation is

$$0 = \det(A - \lambda I) = -\lambda(-c - \lambda) + k = \lambda^2 + c\lambda + k \qquad \Longrightarrow \qquad \lambda = \frac{-c \pm \sqrt{c^2 - 4k}}{2}$$

so that the roots are complex conjugate, repeated or real and distinct depending on whether $c^2 - 4k$ is negative, zero, or positive, resp. In the ck-plane



If the roots are complex, the spring system is underdamped and the solution from either condition oscillates infinitely often. If the roots are repeated the system is critically damped. If the roots are real distinct, they are both negative and the system is overdamped. In the critically and overdamped cases, the solution may overshoot $x = 0$ at most once. However, with the given initial conditions, in these cases the solution returns to $x = 0$ monotonically.

e.g., for an overdamped example $c = 5$ and $k = 4$. then $\lambda = -4, -1$ so the general solution is

$$X(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

Thus with initial conditions $X(0) = \binom{1}{0}$ or $\binom{0}{1}$ the solutions are

$$X(t) = \frac{4}{3} e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \frac{1}{3} e^{-4t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}; \qquad X(t) = \frac{1}{3} e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \frac{1}{3} e^{-4t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

For the critically damped example $c = 4$ and $k = 4$. then $\lambda = -2, -2$ so the general solution is

$$X(t) = c_1 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} t \\ 1-2t \end{pmatrix}$$

Thus with initial conditions $X(0) = \binom{1}{0}$ or $\binom{0}{1}$ the solutions are

$$X(t) = e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + 2e^{-2t}\begin{pmatrix} t \\ 1-2t \end{pmatrix}; \qquad X(t) = e^{-2t}\begin{pmatrix} t \\ 1-2t \end{pmatrix}.$$

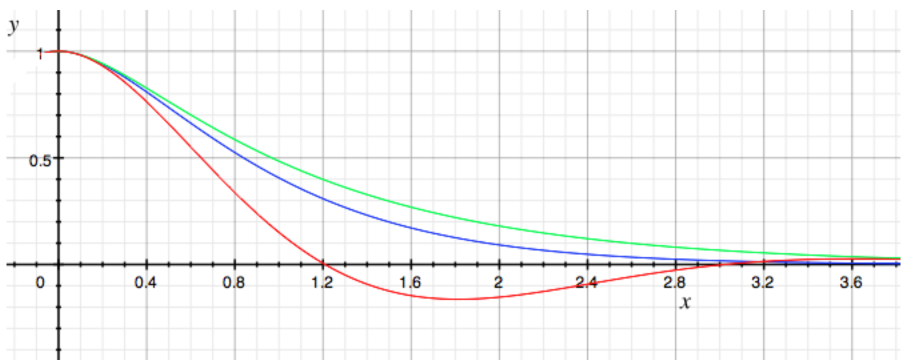
For the overdamped example $c = 2$ and $k = 4$. then $\lambda = -1 \pm \sqrt{3}i$ so the general solution is

$$X(t) = c_1 e^{-t}\begin{pmatrix} \cos(\sqrt{3}t) \\ -\cos(\sqrt{3}t) - \sqrt{3}\sin(\sqrt{3}t) \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin(\sqrt{3}t) \\ \sqrt{3}\cos(\sqrt{3}t) - \sin(\sqrt{3}t) \end{pmatrix}$$

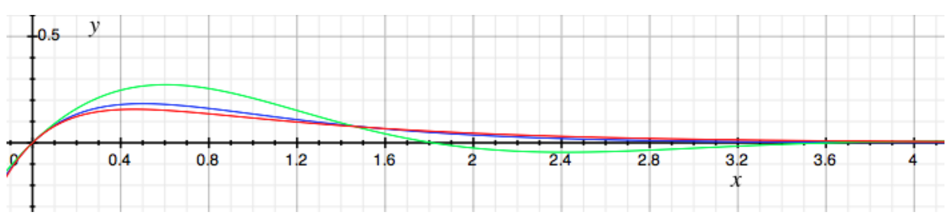
Thus with initial conditions $X(0) = \binom{1}{0}$ or $\binom{0}{1}$ the solutions are

$$X(t) = e^{-t}\begin{pmatrix} \cos(\sqrt{3}t) + \frac{1}{\sqrt{3}}\sin(\sqrt{3}t) \\ -(\sqrt{3} + \frac{1}{\sqrt{3}})\sin(\sqrt{3}t) \end{pmatrix}; \qquad X(t) = \frac{1}{\sqrt{3}}e^{-t}\begin{pmatrix} \sin(\sqrt{3}t) \\ \sqrt{3}\cos(\sqrt{3}t) - \sin(\sqrt{3}t) \end{pmatrix}.$$

The graphs of the solutions beginning from $x = 1$ and $x' = 0$ for over/critically/under damped are green/blue/red as spring constant is constant $k = 4$ and the drag is reduced $c = 5$ to 4 to 2.



For the same constants, the graphs of solutions beginning with $x = 0$ and $x' = 1$ for over/critically/under damped are different colors red/blue/green



9. Find the general solution. Show that there is a unique periodic solution. Find the Poincaré Map $\wp : \{t = 0\} \to \{t = 2\pi\}$ and use it to verify again that there is a unique 2π - periodic solution.

$$x' - x = \cos t.$$

Using integrating factors we find

$$\left(e^{-t}x\right)' = e^{-t}(x' - x) = e^{-t}\cos t$$

so

$$x(t) = e^t x_0 + \int_0^t e^{t-s}\cos s\, ds = e^t x_0 + \frac{1}{2}(e^t - \cos t + \sin t)$$

In order that this be periodic, we need exponential terms to cancel, which happens if $x_0 = -\frac{1}{2}$. To see this solution is unique, since e^t is nonzero we may suppose that it has the form

$$y(t) = e^t f(t) + \frac{1}{2}(e^t - \cos t + \sin t).$$

Since it satisfies the ODE we compute

$$0 = y' - y - \cos t = e^t f'(t)$$

from which follows $f' = 0$ so $f(t) = c$, a constant. Thus all solutions have this form.

The Poincaré map $\wp(x_0)$ is the value of the solution with $x(0) = x_0$ at time $t = 2\pi$. Thus

$$\wp(x_0) = e^{2\pi}x_0 + \frac{1}{2}\left(e^{2\pi} - \cos(2\pi) + \sin(2\pi)\right) = e^{2\pi}x_0 + \frac{1}{2}(e^{2\pi} - 1)$$

A solution is 2π periodic iff $x_0 = \wp(x_0)$. Solving this equation we find the only solution is

$$x_0 = -\frac{1}{2}$$

which corresponds to the only periodic solution.

10. *Find the general solution of $X' = AX$. Assume there is a matrix T such that*

$$T = \begin{pmatrix} 1 & -1 \\ -\frac{3}{2} & 1 \end{pmatrix}; \qquad T^{-1}AT = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}.$$

We see that the matrix T diagonalizes the system. The columns of T are eigenvectors with corresponding diagonal elements of $T^{-1}AT$ the eigenvalues , so that the general solution is

$$X(t) = c_1 e^{-3t}\begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

11. *Let A be a 2×2 real matrix. Suppose $f(x) = Ax : \mathbf{R}^2 \to \mathbf{R}^2$ is onto. Then A is invertible.*

To show that A is invertible, we shall construct a matrix such that $AB = I$. Since $f(x) = Ax$ is onto, there are vectors V_1 and V_2 such that $AV_1 = \binom{1}{0}$ and $AV_2 = \binom{0}{1}$. Let B be the matrix whose columns are V_1 and V_2. Thus

$$AB = A(V_1; V_2) = (AV_1; AV_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that $B = A^{-1}$. To see this, we have $\det(AB) = \det(A)\det(B) = \det(I) = 1$ so that $\det(A) \neq 0$ and so A^{-1} can be given by the formula

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \text{where} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

By premultiplying $AB = I$ by A^{-1} we find $A^{-1}AB = IB = B = A^{-1}$.

12. Let A and B be 2×2 real matrices. Show $\det(AB) = \det(A)\det(B)$.

This one is just a computation for the 2×2 case. Letting

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

we have

$$\det(A) = ad - bc; \qquad \det(B) = eh - fg$$

so

$$\det(A)\det(B) = (ad - bc)(eh - fg) = adeh - adfg - bceh + bcfg.$$

On the other hand

$$AB = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

Thus

$$\begin{aligned} \det(AB) &= (ae + bg)(cf + dh) - (af + bh)(ce + dg) \\ &= acef + adeh + cbfg + bdgh - acef - adfg - bceh - bdgh \\ &= adeh + bcfg - adfg - bceh \end{aligned}$$

so is equal to $\det(A)\det(B)$.