Math 5410 § 1. **Second Midterm Exam** Name: Solutions
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1. *Find* e^{tA}.

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

First method to transform to canonical form. The characteristic equation is

$$0 = \begin{vmatrix} -\lambda & 1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda^3$$

so $\lambda = 0, 0, 0$. Computing eigenvectors and cyclic vectors we find

$$0 = (A - \lambda I)v_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v_1 = (A - \lambda I)v_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$v_2 = (A - \lambda I)v_3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Put

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \qquad T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad J = T^{-1}AT = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$e^{tA} = e^{tTJT^{-1}} = Te^{tJ}T^{-1} = T \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} T^{-1} = \begin{pmatrix} 1 & t + \frac{t^2}{2} & t \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix}.$$

Second method is to compute the power series. Indeed,

$$A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A^3 = 0,$$

so the exponential series terminates at the quadratic term, yielding

$$e^{tA} = I + tA + \frac{t^2}{2}A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t + \frac{t^2}{2} & t \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix}.$$

2. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

(a) STATEMENT: *The set of matrices A that don't have π as an eigenvalue are open and dense in the set of real matrices.*

TRUE. Let U be the set of real $n \times n$ matrices whose eigenvalues all differ from π. The key fact is that the eigenvalues depend continuously on the matrix. A slick way to say that all eigenvalues are not π is

$$f(A) = \prod_{k=1}^{n} (\lambda_i(A) - \pi) \neq 0.$$

Thus U is open because it is the preimage under a continuous function of an open set, namely,

$$U = f^{-1}\big((-\infty, 0) \cup (0, \infty)\big).$$

Equivalently one could say since $\lambda_i(A)$ is continuous, for any $A \in U$, all eigenvalues are some positive distance ϵ away from π, so for sufficiently small $\delta > 0$, if any matrix B satisfies $|B - A| < \delta$ then all $|\lambda_i(A) - \lambda_i(B)| < \epsilon$ so all $\lambda_i(B) \neq \pi$. Thus every matrix in U has a δ-neighborhood of matrices entirely contained in U. Thus U is open.

To see that U is dense, one has to prove that every matrix $A \in L(\mathbf{R}^n)$ can be arbitrarily closely approximated by a matrix in U. But one can choose a sequence $t_i \downarrow 0$ decreasing to zero and consider the approximating matrices $A_i = A + t_i I$ whose eigenvalues are $\lambda_i + t_i$ (why?) For all but finitely many i, the eigenvalues of A_i are not π so $A_i \in U$ and $|A_i - A| \to 0$ as $i \to \infty$. Thus U is dense.

(b) STATEMENT: *Let $a, b \in \mathbb{N}$ be positive integers. Then the solution $(x(t), y(t))$ of the harmonic oscillator equations $\ddot{x} + ax = 0$, $\ddot{y} + by = 0$ is periodic.*

FALSE. Writing in polar coordinates $x = r_1(t)\cos\theta_1(t)$, $\dot{x} = r_1(t)\sin\theta_1(t)$, $y = r_2(t)\cos\theta_2(t)$, $\dot{y} = r_2(t)\sin\theta_2(t)$, the system reduces to $\dot{\theta}_1 = -\sqrt{a}$ and $\dot{\theta}_2 = -\sqrt{b}$. The solutions of this system of oscillators is periodic if and only if the trajectory of $(\theta_1(t), \theta_2(t))$ closes up in the two torus (the square $[0, 2\pi) \times [0, 2\pi) \subset \mathbf{R}^2$ with sides identified). This happens if and only if the ratio of angular frequencies $\sqrt{b}/\sqrt{a}$ is rational. However, if one chooses $a = 4$ and $b = 5$ then this ratio is irrational and the solution is not periodic. The $x(t)$ and $y(t)$ are "out of sync."

(c) STATEMENT: *If $f : \mathbf{R} \to \mathbf{R}$ is a continuously differentiable, then the IVP $\dot{x} = f(x)$ and $x(0) = 0$ has a solution $x(t)$ defined for $t \in \mathbf{R}$.*

FALSE. The solution of $\dot{x} = f(x)$ and $x(0) = 0$ may not exist for all of $t \in \mathbf{R}$. For example $f(x) = 1 + x^2$ is continuosly differentiable but the solution of the IVP is $x(t) = \tan t$ which exists only for $-\frac{\pi}{2} < t < \frac{\pi}{2}$, and tends to infinity as $t \to \pm\frac{\pi}{2}$.

3. *Solve the initial value problem*

$$X' = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} X + \begin{pmatrix} 0 \\ 2e^t \end{pmatrix}, \qquad X(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

We have a real canonical form with eigenvalues $\lambda = 1 \pm 2i$ with

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \qquad e^{tA} = e^t \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix}, \qquad f(t) = \begin{pmatrix} 0 \\ 2e^t \end{pmatrix}.$$

Using the variation of constants formula,

$$\begin{aligned}
X(t) &= e^{tA} \left(X(0) + \int_0^t e^{-sA}\, f(s)\, ds \right) \\
&= e^t \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix} \left\{ \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \int_0^t e^{-s} \begin{pmatrix} \cos 2s & -\sin 2s \\ \sin 2s & \cos 2s \end{pmatrix} \begin{pmatrix} 0 \\ 2e^s \end{pmatrix} ds \right\} \\
&= e^t \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix} \left\{ \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \int_0^t \begin{pmatrix} -2\sin 2s \\ 2\cos 2s \end{pmatrix} ds \right\} \\
&= e^t \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix} \left\{ \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \begin{pmatrix} \cos 2t - 1 \\ \sin 2t \end{pmatrix} \right\} \\
&= e^t \begin{pmatrix} 1 + 2\cos 2t + 5\sin 2t \\ 5\cos 2t - 2\sin 2s \end{pmatrix}
\end{aligned}$$

4. *Suppose* $x_0 \in \mathbf{R}$ *and* $x : \mathbf{R} \to \mathbf{R}$ *is a continuous function that satisfies the equation.*

$$x(t) = x_0 + \int_0^t \sin(s + x(s))\, ds.$$

Why is $x(t)$ *continuously differentiable? State the initial value problem satisfied by* $x(t)$*. Estimate the magnitude of* $x(t)$ *as a function of* t*. For a continuous function* $y : \mathbf{R} \to \mathbf{R}$*, let*

$$J[y](t) = x_0 + \int_0^t \sin(s + y(s))\, ds.$$

Let $y_0(t) = x_0$ *and* $y_{n+1}(t) = J[y_n](t)$*. Is* $\{y_n(t)\}$ *convergent for* $t \in \mathbf{R}$*? Is it convergent on* $t \in [0, \frac{1}{2}]$*?* *Hint:* $|\sin z - \sin w| \le |z - w|$.

Since we assume that $x(t)$ is continuous, $\sin(s + x(s))$ is a continuous function of s. So $x(t)$ is the definite integral of a continuous function, thus continuously differentiable. Differentiating using the Fundamental Theorem of Calculus, and evaluating at $t = 0$,

$$\begin{aligned} \dot{x}(t) &= \sin(t + x(t)), \\ x(0) &= x_0. \end{aligned}$$

The estmate is the same one used to show that the solution of the integral equation stays inside a rectangle. Namely, because $|\sin(s + x(s))| \le 1$ for any $x(t)$ and s, we have for any $t \ge 0$,

$$\begin{aligned} |x(t)| &= \left| x_0 + \int_0^t \sin(s + x(s))\, ds \right| \\ &\le |x_0| + \int_0^t |\sin(s + x(s))|\, ds \\ &\le |x_0| + \int_0^t 1\, ds \\ &\le |x_0| + t. \end{aligned}$$

We get similarly $|x(t)| \le |x_0| - t$ for any $t \le 0$. Putting these together, $|x(t)| \le |x_0| + |t|$ for all $t \in \mathbf{R}$.

The Picard Sequence $y_0(t) = x_0$ and $y_{n+1}(t) = J[y_n](t)$ converges if we can show the sequence $\{y_n(t)\}$ is a Uniformly Cauchy Sequence on $\mathbf{R}$ or on $[0, \frac{1}{2}]$. This will follow if we can show $\|y_{n+1}(t) - y_n(t)\|_0 \le \frac{1}{2}\|y_n - y_{n-1}\|_0$ for $n \ge 1$ in the continuous functions $\mathcal{C}(\mathbf{R})$ or in $\mathcal{C}([0, \frac{1}{2}])$. Recall that $\|f\|_0 = \sup\{|f(t)| : t \in \text{domain.}\}$. For the first step, we have

$$|y_1(t) - y_0(t)| = \left| \int_0^t \sin(s + y_0)\, ds \right| = |1 - \cos(t + y_0)| \le 2$$

for all $t \in \mathbf{R}$. Using the hint $|\sin z - \sin w| \le |z - w|$, that $\sin(x)$ is 1-Lipschitz, we estimate the case $t \geq 0$ for simplicity

$$\begin{aligned} |y_{n+1}(t) - y_n(t)| &= \left| \int_0^t \sin(s + y_n(s))\, ds - \int_0^t \sin(s + y_{n-1}(s))\, ds \right| \\ &\leq \int_0^t |\sin(s + y_n(s)) - \sin(s + y_{n-1}(s))|\, ds \\ &\leq \int_0^t |s + y_n(s) - s - y_{n-1}(s)|\, ds \\ &= \int_0^t |y_n(s) - y_{n-1}(s)|\, ds \\ &\leq \int_0^t \|y_n - y_{n-1}\|_0\, ds \\ &= t\|y_n - y_{n-1}\|_0. \end{aligned}$$

This is not a bounded quantity if t is allowed to be unbounded as for $t \in \mathbf{R}$. Thus this estimate fails in $\mathbf{R}$ case. However, if $0 \leq t \leq \frac{1}{2}$ we get

$$|x_{n+1}(t) - x_n(t)| \leq \frac{1}{2}\|x_n - x_{n-1}\|_0$$

Taking sup over $[0, \frac{1}{2}]$ we get

$$\|x_{n+1} - x_n\|_0 \leq \frac{1}{2}\|x_n - x_{n-1}\|_0$$

in $\mathcal{C}([0, \frac{1}{2}])$. Thus, with a little work we can deduce that $\{x_n\}$ is a Cauchy Sequence in $\mathcal{C}([0, \frac{1}{2}])$ and so converges to a solution of the integral equation.

5. *Find the first few Picard iterates of the system. Show that they converge to a solution of the IVP.*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

First find the solution of the IVP. The first equation is independent of the second.

$$\dot{x} = 1, \qquad x(0) = 1.$$

Integrating, its solution is $x(t) = 1 + t$. Then the second equation becomes

$$\dot{y} = x = 1 + t, \qquad y(0) = 1.$$

Its solution is $y(t) = 1 + t + \frac{t^2}{2}$.

Let's do Picard Iteration. It can start with any arbitrary continuous $Z_0 \in \mathcal{C}([0, \frac{1}{2}])$, so we choose $Z_0(t) = \binom{3}{4}$.

$$\begin{aligned} Z_0(t) &= \begin{pmatrix} 3 \\ 4 \end{pmatrix} \\ Z_1(t) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^t F(Z_0(s))\, ds = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 \\ 3 \end{pmatrix} ds = \begin{pmatrix} 1 + t \\ 1 + 3t \end{pmatrix} \\ Z_2(t) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^t F(Z_1(s))\, ds = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 \\ 1 + s \end{pmatrix} ds = \begin{pmatrix} 1 + t \\ 1 + t + \frac{t^2}{2} \end{pmatrix} \\ Z_3(t) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^t F(Z_2(s))\, ds = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 \\ 1 + s \end{pmatrix} ds = \begin{pmatrix} 1 + t \\ 1 + t + \frac{t^2}{2} \end{pmatrix} \end{aligned}$$

The sequence stabilizes $Z_2(t) = Z_3(t) = Z_4(t) = \cdots$ and has converged in two steps to the solution of the system.