Homework for Math 5410 §1, Fall 2020

A. Treibergs, Instructor

December 2, 2020

Our text is by Morris Hirsch, Stephen Smale & Robert Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos* 3rd ed., Academic Press, 2013. Please read the relevant sections in the text as well as any cited reference. Assignments are due the following Friday, or on Nov. 30, whichever comes first.

Your written work reflects your professionalism. Make answers complete and self contained. This means that you should copy or paraphrase each question, provide adequate explanation to help the reader understand the structure of your argument, be thorough in the details, state any theorem that you use and proofread your answer.

Homework from Wednesday to Tuesday will be due Friday. Homework is to be uploaded to Canvas by 4:00 pm, Fridays to be considered on time. Late homework that is up to one week late will receive half credit. Homework that is more than one week late will receive no credit at all.

Please hand in problems A on Friday, August 28.

A. Exercises from the text by Hirsch, Smale & Devaney:

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18[1-4, 9, 14]
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Please hand in problems B on Friday, Sept. 4.

B. Exercises from the text by Hirsch, Smale & Devaney:

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37[2b, 7, 9, 10, 11, 14]
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Please hand in problems C on Friday, Sept. 8.

C. Exercises from the text by Hirsch, Smale & Devaney:

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57[4, 5, 7, 10, 11,  14]
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Please hand in problems D on Friday, Sept. 19.

D. Exercises from the text by Hirsch, Smale & Devaney:

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71[1, 2, 3, 4, 5]
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Please hand in problems E on Friday, Sept. 25.

E. Exercises from the text by Hirsch, Smale & Devaney:

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103[4, 7, 8, 11, 12]
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Please hand in problems F on Friday, Oct. 2.

F. Exercises from the text by Hirsch, Smale & Devaney:

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103[5(any three)]
The rest of the problems are postponed to next week.
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Please hand in problems G on Friday, Oct. 9.

Reminder: your project outlines are due Oct. 23.

G. Exercises from the text by Hirsch, Smale & Devaney:

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103[13, 14, 15]
135[1(any three), 4, 9]
The rest of the problems are postponed to next week.
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Please hand in problems H1 – H2 on Friday, Oct. 16.

Reminder: your project outlines are due Oct. 23.

H1. Exercises from the text by Hirsch, Smale & Devaney:

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135[12aej, 13]
157[1a, 2]
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H2. Solve the initial value problem:

$$\frac{d}{dt}X = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} X + \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}; \qquad X(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Please hand in problems I1 – I3 on Friday, Oct. 23.

Reminder: your project outlines are due Oct. 23.

I1. Exercises from the text by Hirsch, Smale & Devaney:

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157[5]
407[9]
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I2. Show that the iteration scheme $\psi_0(t) = A$,

$$\psi_{n+1}(t) = A + Bt + \int_0^t (s - t)\psi_n(s)\, ds$$

will converge to a solution of the problem $\ddot{x} + x = 0$, $x(0) = A$, $\dot{x}(0) = B$ for certain values of t. For what values of t is convergence assured? [From H. K. Wilson, Ordinary Differential Equations, Addison-Wesley, 1971, p.245.]

I3. The Contraction Mapping Principle. Here is the abstract idea behind the Picard Theorem. Let $V \subset \mathbf{R}^n$ be a closed subset. Let $0 < b < \infty$ and $0 < k < 1$ be constants and let $T : V \to V$ be a transformation. Suppose that for any $\phi,\ \psi \in V$ if $|\psi| \le b$ then $|T(\psi)| \le b$ and if both $|\phi| \le b$ and $|\psi| \le b$ then

$$|T(\psi) - T(\phi)| \le K|\psi - \phi|,$$

i.e., T is a *contraction.* Prove that there exists an element $\eta \in V$ with $|\eta| \le b$ such that $\eta = T(\eta)$, that is, T has a fixed point. Prove that η is the unique fixed point among points in V satisfying $|\eta| \le b$. [Coddington & Levinson, Theory of Ordinary Differential Equations, Krieger 1984, pp. 40-41.]

Please hand in problems J1 – J4 on Friday, Oct. 30.

J1. Let a, b and p be positive constants. Consider the differential equation

$$\begin{aligned}\dot{x} &= -\frac{ax}{\sqrt{x^2 + y^2}}\\ \dot{y} &= -\frac{ay}{\sqrt{x^2 + y^2}} + b\end{aligned}$$

which models the flight of a bird heading toward the origin at constant speed a, that is moved off course by a steady wind of velocity b. Determine the conditions on a and b to ensure that the solution starting at $(p, 0)$ reaches the origin. Hint: change to polar coordinates and study the phase portrait of the differential equation on the cylinder. [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 86.]

J2. Prove the Generalized Gronwall Inequality: Suppose $a(t)$, $b(t)$ and $u(t)$ are continuous functions defined for $0 \le t < \infty$ and that $b(t) \ge 0$ for all $t \ge 0$. Suppose that

$$u(t) \le a(t) + \int_0^t b(s)\, u(s)\, ds, \qquad \text{for all } t \ge 0.$$

Show that

$$u(t) \le a(t) + \int_0^t a(s)\, b(s) \exp\left(\int_s^t b(\tau)\, d\tau\right) ds, \qquad \text{for all } t \ge 0.$$

J3. Find a sharp estimate for the difference in values and derivatives at T of the solutions for the two initial value problems, where u_0, u_1, ϵ are constants.

$$\begin{aligned} \ddot{x} + x &= 0, \\ x(0) &= u_0, \\ \dot{x}(0) &= u_1; \end{aligned} \qquad\qquad \begin{aligned} \ddot{y} + (1 + \epsilon \sin(3t))y &= 0, \\ y(0) &= u_0, \\ \dot{y}(0) &= u_1. \end{aligned}$$

J4. Let

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{pmatrix}.$$

Find the solution $\varphi(t, y) \in \mathbf{R}^3$ of

$$\begin{aligned} \frac{dx}{dt} &= f\big(x(t)\big), \\ x(0) &= y. \end{aligned}$$

Find

$$\Phi(t, y) = D_2\, \varphi(t, y).$$

Show that it satisfies the variational equation

$$\begin{aligned} \frac{d\Phi}{dt} &= Df\big(\varphi(t, y)\big) \cdot \Phi(t, y), \\ \Phi(0) &= I. \end{aligned}$$

[Perko, p. 84.]

Please hand in problems K1–K2 on Friday, Nov. 6.

K1. Exercises from the text by Hirsch, Smale & Devaney:

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184[1, 2, 5, 8]
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K2. Linearized Stability of Fixed Points.

The SIR model of epidemics of Brauer and Castillo-Chávez relates three populations, $S(t)$ the susceptible population, $I(t)$ the infected population and $R(t)$ the recovered population. The other variables are positive constants. Assume that births in the susceptible group occur at a constant rate μK. Assume that there is a death rate of $-\mu$ for each population. Assume also that there is an infection rate of people in the susceptible population who become infected which is proportional to the contacts between the two groups βSI. There is a recovery of γI from the infected group into the recovered group. Finally, the disease is fatal to some in the infected group, which results in the removal rate $-\alpha I$ from the infected population. The resulting system of ODE's is

$$\begin{aligned} \dot{S} &= \mu K - \beta SI - \mu S \\ \dot{I} &= \beta SI - \gamma I - \mu I - \alpha I \\ \dot{R} &= \gamma I - \mu R \end{aligned}$$

1. Note that the first two equations decouple and can be treated as a 2×2 system. Then the third equation can be solved knowing $I(t)$. Let $\delta = \alpha + \gamma + \mu$. For the 2×2 system, find the nullclines and the fixed points.
2. Check the stability of the nonnegative fixed points. Show that for $\beta K < \delta$ the disease dies out. Sketch the nullclines and some trajectories in the phase plane in this case.
3. Show that for $\beta K > \delta$ the epidemic reaches a steady state. Sketch the nullclines and some trajectories in the phase plane now.

[From R. C. Robinson, *An Introduction to Dynamical Systems*, Pearson 2004.]

Please hand in problems L1 on Friday, Nov. 13.

L1. Exercises from the text by Hirsch, Smale & Devaney:

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210[1(any two of a-e), 2, 4, 6]
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Please hand in problems M1 – M2 on Friday, Nov. 20.

M1. Exercises from the text by Hirsch, Smale & Devaney:

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210[7(any two), 8(any two), 11]
229[1(any two), 2]
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M2. Determine the stability types at the origin for the following systems.

$$(a.) \quad \begin{cases} x' = -x^3 + xy^2 \\ y' = -2x^2y - y^3 \end{cases} \qquad (b.) \quad \begin{cases} x' = -x^3 + 2y^3 \\ y' = -2xy^2 \end{cases}$$

$$(c.) \quad \begin{cases} x' = x^3 - y^3 \\ y' = xy^2 + 2x^2y + y^3 \end{cases} \qquad (d.) \quad \begin{cases} x' = x^3 + xy \\ y' = -y + y^2 + xy - x^3 \end{cases}$$

[J. Hale and H. Koçak, *Dynamics and Bifurcations*, Springer 1991, p. 285.]

You don't need to do problems N1 – N6. This material will be on the final.

These problems will be graded if you choose to hand them in by Friday, Dec. 11.

Please hand in Term Project on Wednesday, Dec. 2.

N1. Exercises from the text by Hirsch, Smale & Devaney:

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228[7, 10, 16]
255[10]
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N2. Show that the system has a nontrivial periodic orbit.

$$\dot{x} = y$$
$$\dot{y} = -x + y(9 - 4x^2 - y^2)$$

N3. The Maxwell-Bloch equations for a laser are

$$\dot{E} = \kappa(P - E)$$
$$\dot{P} = \gamma_1(ED - P)$$
$$\dot{D} = \gamma_2(\lambda + 1 - D - \lambda EP)$$

1. Show that the non-lading state (the fixed point with $E^* = 0$) loses stability above a threshold value of λ, to be determined. Classify the bifurcation at this laser threshold.
2. Find the change of variables that transforms this system into the Lorenz system.

N4. Show that all trajectories of the Lorenz equations eventually enter and remain inside the sphere

$$x^2 + y^2 + (z - r - \sigma)^2 = C$$

for C sufficiently large. [Hint: show that $x^2+y^2+(z-r-\sigma)^2$ decreases on trajectories outside a certain ellipsoid. Then pick C large enough so that the sphere engulfs this ellipsiod.]

N5. Consider Rikitake's model for geommagnetic reversals with $a, \nu > 0$ parameters

$$\dot{x} = -\nu x + zy$$
$$\dot{y} = -\nu y + (z - a)x$$
$$\dot{z} = 1 - xy$$

1. Show that the system is dissipative.
2. Show that the fixed points may be written $x^* = \pm k$, $y^* = \pm k^{-1}$, $z^* = \nu k^2$ where $\nu(k^2 - k^{-2}) = a$.
3. Classify the fixed points.

N6. Consider the following system in polar coordinates. $\dot{r} = r(1 - r^2)$, $\dot{\theta} = 1$.

Let $D = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\}$ be the closed unit disk.

1. Is D an invariant set?
2. Does D attract an open set of initial conditions?
3. Is D an attractor? If not, why not? If so, find its basin of attraction.
4. Repeat part (c) for the circle $C = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$.

[**N3** − **N6** are from Strogatz *Nonlinear Dynamics and Chaos,* Westview, Boulder, 1994, pp.342–344.]

The **FINAL** for Math 5410 - 1 is Friday, Dec. 11 from 8:00–10:00 am. Half of the final will be comprehensive. The rest will concentrate on material since the last midterm, sections 8.1–8.5, 9.1–9.5, 10.1–10.7, 11.1–11.3, 14.1–14.3. The final will be an open book canvas quiz.