Math 5410 § 1. **First Midterm Exam** Name: Solutions
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1. *For the differential equation find the general solution. Find the Poincaré Map for 2π periodic solutions. Is there a 2π-periodic solution? Why or why not?*

$$x' = 1 + x\cos t$$

Use integrating factor $e^{-\sin t}$. Thus

$$\frac{d}{dt}\left[e^{-\sin t}x\right] = e^{-\sin t}\left[\dot{x} - (\cos t)x\right] = e^{-\sin t}$$

so

$$e^{-\sin t}x(t) = c_1 + \int_0^t e^{-\sin s}\,ds$$

or

$$x(t) = c_1 e^{-\sin t} + e^{-\sin t}\int_0^t e^{-\sin s}\,ds$$

where $x(0) = c_1$. The Poincaré Map is where the initial point flows to at $t = 2\pi$. Since $\sin 2\pi = 0$, we have

$$\wp(c_0) = c_0 + \int_0^{2\pi} e^{-\sin s}\,ds.$$

There is a fixed point if $\wp(c_0) = c_0$. But since the integral is positive we have $\wp(c_0) > c_0$ for all c_0 so there is no fixed point of the Poincaré Map, thus no 2π-periodic solution.

2. *What is the real canonical form of A? Find a matrix that transforms A to canonical form and show your matrix does the job. [Hint: A has a triple eigenvalue $\lambda = 1, 1, 1$.]*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$

We can see the matrix $A - \lambda I$ has rank one, so there are two eigenvectors we can find by inspection.

$$(A - \lambda I)[V_1, V_3] = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

First Method: solve for a cyclic vector. Notice that there are no solutions to $(A-\lambda I)V_4 = V_3$ so V_3 is in its own 1×1 block. But we may solve for V_2 from $(A-\lambda I)V_2 = V_1$ by inspection

$$(A - \lambda I)V_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Putting $P = (V_1|V_1|V_3)$ we check that $P^{-1}AP = J$ or

$$AP = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = PJ.$$

Second method. Choose W_2 not in the null space of $A - \lambda I$, let $W_1 = (A - \lambda I)W_2$ and then choose another vector in the nullspace independent of W_1 and W_2. We pick W_2 and compute W_1.

$$W_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad W_1 = (A - \lambda I)W_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}$$

Then a null vector independent of W_1 and W_2 is

$$W_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Putting $Q = (W_1|W_1|W_3)$ we check that $Q^{-1}AQ = J$ or

$$AQ = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 3 & 1 & 0 \\ -3 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 3 & 4 & 0 \\ -3 & -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 1 \\ 3 & 1 & 0 \\ -3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = QJ.$$

3. *Find the flows ϕ_t^X and ϕ_t^Y. Find an explicit topological conjugacy between the flows. Check that your conjugacy works.*

$$X' = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} X, \qquad\qquad Y' = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} Y.$$

The flows decouple. They are given by

$$\phi_t^X(c_1, c_2) = (c_1 e^t, c_2 e^{-3t}), \qquad \phi_t^Y(d_1, d_2) = (d_1 e^{-2t}, d_2 e^{2t})$$

The map H flips x_1, x_2 and takes e^t to e^{2t} so is the square map, and takes e^{-3t} to e^{-2t} which is raising to the $\frac{2}{3}$ power. Thus

$$H(x_1, x_2) = \left(\operatorname{sgn}(x_2)\,|x_2|^{3/2}, \operatorname{sgn}(x_1)|x_1|^2\right).$$

We check that this does the job, namely, satisfies the conjugacy equation

$$H\big(\phi_t^X(c)\big) = \phi_t^Y\left(H(c)\right)$$

for all c and t. We see that

$$\begin{aligned} H\big(\phi_t^X(c)\big) &= H\left(c_1 e^t, c_2 e^{-3t}\right) = \left(\operatorname{sgn}(c_2 e^{-3t})\,|c_2 e^{-3t}|^{2/3}, \operatorname{sgn}(c_1 e^t)\,|c_1 e^t|^2\right) \\ &= \left(\operatorname{sgn}(c_2)\,|c_2|^{2/3} e^{-2t}, \operatorname{sgn}(c_1)\,|c_1|^2 e^{2t}\right) \end{aligned}$$

whereas

$$\begin{aligned} \phi_t^Y\left(H(c)\right) &= \phi_t^Y\left(\operatorname{sgn}(c_2)\,|c_2|^{3/2}, \operatorname{sgn}(c_1)|c_1|^2\right) \\ &= \left(\operatorname{sgn}(c_2)\,|c_2|^{2/3} e^{-2t}, \operatorname{sgn}(c_1)\,|c_1|^2 e^{2t}\right). \end{aligned}$$

As these are the same expression, this H establishes the flow conjugacy.

4. *For the system determine the canonical form for this equation. Find the matrix P so that $Y = PX$ puts (1) in canonical form. Check that your matrix works. Find the real valued general solution. Find the flow $\phi_t^X(c)$ generated by this equation.*

$$X' = \begin{pmatrix} 1 & -4 \\ 2 & 5 \end{pmatrix} X. \tag{1}$$

The eigenvalues of the matrix A satisfy

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -4 \\ 2 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda) + 8 = \lambda^2 - 6\lambda + 13 = (\lambda - 3)^2 + 4.$$

Thus $\lambda = 3 \pm 2i$. The canonical form of the matrix A is the matrix

$$J = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}.$$

A complex eigenvector for $\lambda = 3 + 2i$ may be seen by inspection

$$(A - \lambda I)W = \begin{pmatrix} -2-2i & -4 \\ 2 & 2-2i \end{pmatrix} \begin{bmatrix} 1-i \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Select the real and imaginary parts

$$V_1 = \Re\mathrm{e}\, W = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad V_2 = \Im\mathrm{m}\, W = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Putting $P = (V_1|V_1)$ we check that $P^{-1}AP = J$ or

$$AP = \begin{pmatrix} 1 & -4 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} = PJ.$$

We solve the canonical equation

$$Y' = JY = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} Y, \quad y(0) = c \qquad \Longrightarrow \qquad \phi_t^Y(c) = e^{3t} \begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos 2t \end{pmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Then in the original variables by $X = PY$ so the general solution is

$$\begin{aligned} X(t) = P\phi_t^Y(c) &= e^{3t} \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos 2t \end{pmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= c_1 e^{3t} \begin{bmatrix} \cos 2t + \sin 2t \\ -\cos 2t \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} \sin 2t - \cos 2t \\ -\sin 2t \end{bmatrix}. \end{aligned}$$

5. *Consider the family of differential equations depending on the parameter a. Find the bifurcation points. Sketch the bifurcation diagram for this family of equations. Identify the rest points on the bifurcation diagram as sources, sinks or neither. Sketch the phase lines for values of a above and below the bifurcation values.*

$$x' = x^3 + 2x^2 - ax$$

The bifurcation curve of the ODE

$$x' = x^3 + 2x^2 - ax = f(x, a)$$

is the locus of $f(x, a) = 0$, namely solutions of

$$0 = f(x, a) = x\left(x^2 + 2x - a\right)$$

which is the union of locii of $x = 0$ or $a = x^2 + 2x = x(x + 2)$. Two bifurcations occur. The first at $(a, x) = (-1, -1)$ where a rest point is created and two rest points separate from $x = -1$ as a increases through -1. One of the rest points is a sink (S) and the other is a source (U). Tbis one is called a "saddle-node" bifurcation. The second is at $(0, 0)$ where two rest points cross as a increases through 0. This one is called a "transcritical" bifurcation. We find the sign of $f(x, a)$ to see if x is increasing. No matter what a is, $f > 0$ when x is very positive and $f < 0$ when x is very negative.

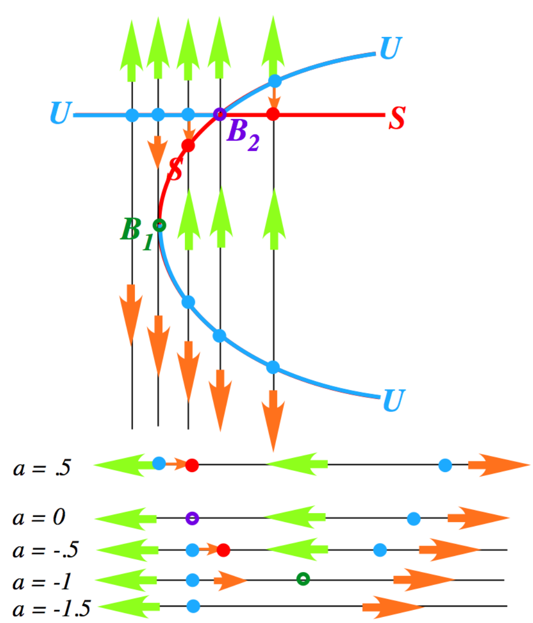

Figure 1: Bifurcation Diagram and Phase Lines at Various a's