Math 5410 § 1. **Third Midterm Exam** Name:____________________
Treibergs Nov. 17, 2021

1. *Find the values of the parameter a where bifurcations occur. Describe the nature of the bifurcations. Sketch the phase plane for three a's, one at a bifurcation value one just below and the other just above.*

$$\dot r = r - r^3$$
$$\dot\theta = a + \cos\theta$$

$\dot r = 0$ when $r = 0$ and $r = 1$. $\dot r > 0$ in $0 < r < 1$ making 0 an unstable equilibrium point for all a. Also $\dot r < 0$ if $r > 1$ making the circle $C = \{r = 1\}$ an attracting invariant set for all a. Different values of a affect the direction of flow on C.

The bifurcation curve in the a-θ plane is the solution of the equation $0 = a + \cos\theta$, in other words $a = -\cos\theta$, This means that there are $\dot\theta = 0$ points for $-1 \le a \le 1$ and no rest points otherwise. In fact there are no rest points for $a > 1$ one rest point at $(a, \theta) = (1, \pi)$ for *theta* $\in [0, 2\pi)$ which is attrctive but not Liapunov stable, two rest points (a, θ_1), a stable node and $(a, 2\pi - \theta_1)$ an unstable saddle for $-1 < a < 1$ where $0 < \theta_1 < \pi$ satisfies $\cos(\theta_1) = -a$, one rest point $(-1, 0)$ which is attractive but not Liapunov stable when $a = -1$ and no rest points if $a > 1$. Thus there are two saddle/node bifurcation points, at $(-1, 0)$ and $(1, \pi)$.

For various a's the θ phase line $[0, 2\pi]$ look like

$a > 1$		$\longrightarrow$		$\longrightarrow$		$\longrightarrow$		$\longrightarrow$	
$a = 1$		$\longrightarrow$		$\longrightarrow$	π	$\longrightarrow$		$\longrightarrow$	
$-1 < a < 1$		$\longrightarrow$	θ_1	$\longleftarrow$		$\longleftarrow$	$2\pi - \theta_1$	$\longrightarrow$	
$a = -1$	0	$\longleftarrow$		$\longleftarrow$		$\longleftarrow$		$\longleftarrow$	2π
$a < -1$		$\longleftarrow$		$\longleftarrow$		$\longleftarrow$		$\longleftarrow$	

Here are "Slopes" generated plots.

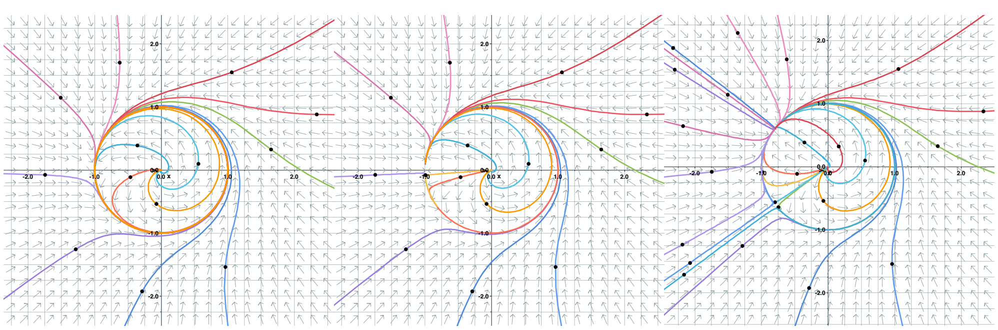

Figure 1: Saddle-node Bifurcation at $a = 1.2$, $a = 1$ and $a = .8$.

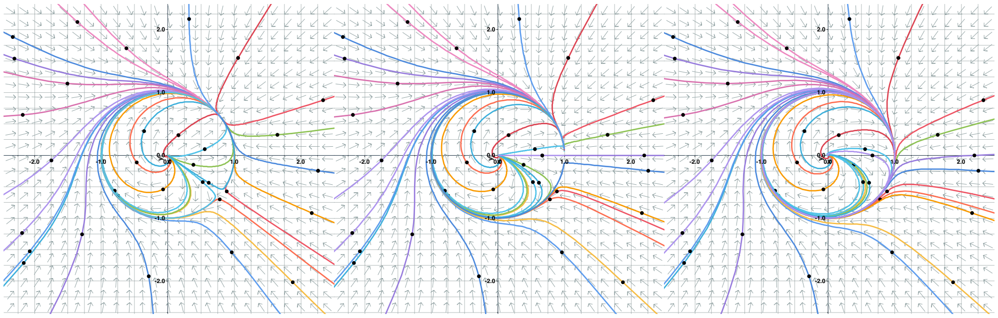

Figure 2: Saddle-node Bifurcation at $a = -.8$, $a = -1$ and $a = -1.2$.

2. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT: *Let the solution $\varphi(t, Y)$ of the smooth planar ODE $X' = F(X)$ and $X(0) = Y$ exist for all $t \in [0, \infty)$. Then the ω-limit set $\omega(Y)$ consists of a single point.*

   FALSE. The $\omega(Y)$ limit set may also be empty or a limit cycle as in $X' = X$ and $Y \neq 0$, or $\dot{r} = r(1 - r^2)$, $\dot{\theta} = 1$ and $(r_0, \theta_0) \neq (0, 0)$. It can also be more complicated such as a heteroclinic cycle.

   (b) STATEMENT: *Let A be a 2×2 matrix and $\varphi(t, Y)$ solves $X' = AX$ and $X(0) = Y$. Then $\Phi(t) = \mathrm{d}_Y\, \varphi(t, Y)$ satisfies $\Phi' = A\Phi$.*

   TRUE. This is just the variation equation for $f(X) = AX$. We can see it two ways. As a variation equation, $\Phi(t) = \mathrm{d}_Y\, \varphi(t, Y)$ satisfies $\Phi' = \mathrm{d}\, f(\varphi(t, Y))\Phi$ but $\mathrm{d}\, f(Z) = AZ$ since it is linear. The other way to see it is to use the fact that $\varphi(t, Y) = e^{tA}Y$ so $\Phi(t) = e^{tA}$ and $\Phi' = Ae^{tA} = A\Phi$.

   (c) STATEMENT: *t $\dot{L}(X) \le 0$ but not $\dot{L}(X) < 0$ for all $X \neq 0$ then 0 is Liapunov Stable but not asymptotically stable.*

   FALSE. If $\dot{L}$ is only nonpositive, the rest point may still be asymptotically stable. An example occurs in the pendulum equation with $b > 0$

$$\begin{cases} \dot{x} = v \\ \dot{y} = -\sin x - by \end{cases}$$

   With $L(x, y) = \frac{1}{2}y^2 + 1 - \cos x$ and $P = \{L(x, y) \le r, |x| < \pi\}$ with $r < 2$ we get $\dot{L} = -by^2$ so only $\dot{L} \le 0$. On the set $Z = \{(x, y) \in P : \dot{L}(x, y) = 0\}$, the only invariant subset is $\{(0, 0)\}$, so by Lasalle's Invariance Principle, $(0, 0)$ is asymptotically stable.

3. *Determine whether the given equilibrium point for the given system is Asymptotically Stable, is Liapunov Stable but Not Asymptotically Stable, or is Not Stable. Give a brief explanation.*

   (a) $x = 0$, $y = 0$ for $\begin{cases} \dot{x} = H_y(x, y) \\ \dot{y} = -H_x(x, y) \end{cases}$ where $H(x, y) = (x + 2y)^2 + (x - y)^4$.

   STABLE BUT NOT ASYMPTOTICALLY STABLE. This is a Hamiltonian System and $H(x, y)$ has a strict minimum at the origin. Since trajectories of Hamiltonian flows stay in the level sets of H, which in this case are concentric ovals that surround $(0, 0)$, the origin is a center which is stable but not attractive.

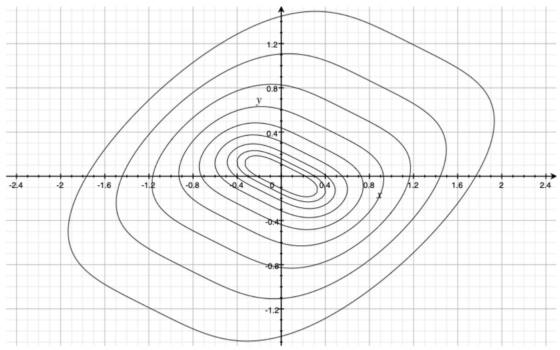

Figure 3: Levels sets $H = .05, .1, .2, .4, .8, 1.6, 3.2, 6.4, 12.8$.

(b) $r = 1$, $\theta = 0$ for $\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = \sin\theta \end{cases}$

Not Stable. Since $\mathrm{d}\, f(r, \theta) = \left(\begin{smallmatrix} 1-2r^2 & 0 \\ 0 & \cos\theta \end{smallmatrix}\right)$ so $\mathrm{d}\, f(1, 0) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, this stationary point is a saddle, which is not stable.

(c) [4] $X = (0, 0)$ for $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Asymptotically stable. Since the matrix has trace $T = -6$ and determinant $D = 5$ with $36 = T^2 > 4D = 20$, the origin of this linear system is a spiral sink, thus is asymptotically stable.

(d) [4] $(x, y) = (0, 0)$ for $\begin{cases} \dot{x} = x^3(y - 1) \\ \dot{y} = y^3(x - 2) \end{cases}$

Asymptotically stable. Consider $L(x, y) = \frac{1}{2}(x^2 + y^2)$, a positive definite function at the origin. Then in the neighborhood of the origin $\Omega = \{(x, y) \in \mathbf{R}^2 : x < 2 \text{ and } y < 1\}$ we have for $(x, y) \in \Omega$ such that $(x, y) \neq (0, 0)$,

$$\dot{V} = x\dot{x} + y\dot{y} = x^4(y - 1) + y^4(x - 2) < 0.$$

Thus this is a strict Liapunov Function and the origin is asymptotically stable.

4. *Consider the solution $x(t)$ of the IVP for $t \geq 0$. Find estimates for $x(t)$ and $\dot{x}(t)$ in terms of t and (u_0, u_1). Does the solution exist for all $t \in [0, \infty)$? Why?*

$$\ddot{x} + \frac{\dot{x}}{1 + x^2} + x = 0, \qquad x(0) = u_0, \quad \dot{x}(0) = u_1.$$

Convert to a 2×2 system.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x - \frac{y}{1+x^2} \end{pmatrix} = F \begin{pmatrix} x \\ y \end{pmatrix}$$

We observe that we have an estimate for $F(X)$ for every $X = \binom{x}{y} \in \mathbf{R}^2$ given by

$$\begin{aligned} |F(X)|^2 = y^2 + \left(-x - \frac{y}{1 + x^2}\right)^2 &= y^2 + x^2 + \frac{2xy}{1 + x^2} + \frac{y^2}{(1 + x^2)^2} \\ &\leq y^2 + x^2 + \frac{x^2 + y^2}{1 + x^2} + \frac{y^2}{(1 + x^2)^2} \leq 3y^2 + 2x^2 \leq 3\left(x^2 + y^2\right) = 3|X|^2. \end{aligned}$$

where we have used $2xy \le x^2 + y^2$.

Now we estimate X using Gronwall's inequality. Assume we have a solution $X(t)$ on $t \in [0, T]$. For every $0 \le t \le T$, the integral equation is

$$X(t) = U_0 + \int_0^t F(X(s))\, ds,$$

where $U_0 = \binom{u_0}{u_1}$. Taking absolute value and using the estimate for $F(X(s))$,

$$|X(t)| \le |U_0| + \int_0^t |F(X(s))|\, ds \le |U_0| + \sqrt{3} \int_0^t |X(s))|\, ds,$$

Gronwall's Inequality gives the desired estimate on X for $0 \le t \le T$,

$$|X(t)| \le |U_0| e^{\sqrt{3}\, t}.$$

The solution exists for all $t \in [0, \infty)$. This follows from the estimate, because if there were a maximal interval of existence $[0, \beta)$ with $\beta < \infty$, then the solution $|X(t)|$ would have had to exit any compact set as $t \to \beta-$. However, the solution satisfies $|X(t)| \le |U_0| e^{\sqrt{3}\, \beta}$ for all $0 \le t < \beta$.

5. *You may assume that the system is defined only for $x, y \ge 0$. The equilibrium points are $(0, 0)$, $(2, 0)$ and $(1, 1)$. Draw the x and y nullclines and use this information to sketch the global phase portrait of this nonlinear system. At the interior equilibrium point $(1, 1)$, give a detailed description of the behavior of the linearized system.*

$$\dot{x} = x(2 - x - y)$$
$$\dot{y} = y(x - 1)$$

The $\dot{x} = 0$ nullclines are the lines $x = 0$ and $x + y = 2$. Flow is vertical on these lines. We have $\dot{x} > 0$ to the left and $\dot{x} < 0$ to the right of $x + y = 2$.

The $\dot{y} = 0$ nullclines are the lines $y = 0$ and $x = 1$. Flow is horizontal on these lines. We have $\dot{y} < 0$ to the left and $\dot{y} > 0$ to the right of $x = 1$.

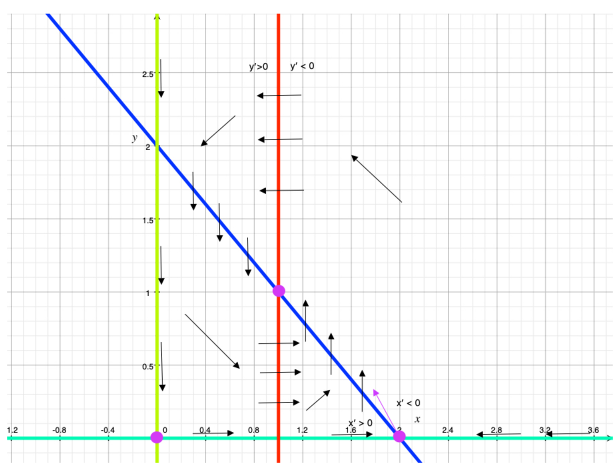

Figure 4: Nullclines and directions of flow for (5).

Hence the direction of flow in the regions cut by the nullclines is SE, NE, NW, SW going anticlockwise in order around $(1, 1)$ starting from the region near the origin.

The Jacobian of $F(X)$ is

$$\mathrm{d}\, F(x,y) = \begin{pmatrix} 2-2x-y & -x \\ y & x-1 \end{pmatrix}$$

At the rest point $(1,1)$ we have

$$\mathrm{d}\, F(1,1) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

It has trace $T=-1$ and determinant $D=1$ with $1=T^2<4D=4$ so that the rest point $(1,1)$ is a spiral sink.

We provide a discussion of the other rest points for completeness sake but it was not required. For the rest point $(0,0)$ we have

$$\mathrm{d}\, F(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

It is a saddle with incoming eigendirection $\binom{0}{1}$ and outgoing eigendirection $\binom{1}{0}$.

At the rest point $(1,1)$ we have

$$\mathrm{d}\, F(2,0) = \begin{pmatrix} -2 & -2 \\ 0 & 1 \end{pmatrix}$$

It is a saddle with incoming eigendirection $\binom{1}{0}$ and outgoing eigendirection $\binom{-2}{3}$.

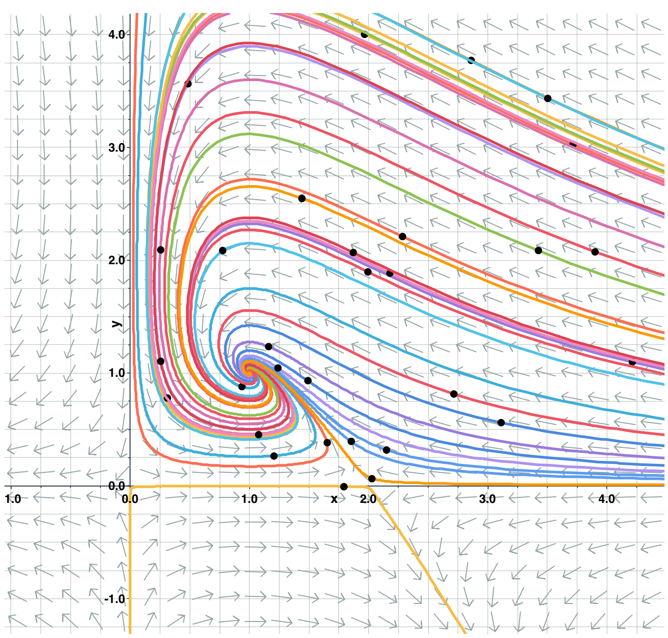

Figure 5: “Slopes” generated phase plane for (5).