Math 5410 § 1. **Second Midterm Exam** Name: Solutions
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1. *Find the general solution. Determine its behavior in* $\mathbf{R}^3$ *as* $t \to \infty$. *[Hint: the eigenvalues are* $-2, -2, -2$.*]*

$$X' = \begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & -1 & -2 \end{pmatrix} X.$$

For $\lambda = -2$ we find the eigenvectors.

$$(A - \lambda I)V_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$A - \lambda I$ has rank two, so there is only one dimension of eigenvectors and there are two independent cyclic vectors associated to the eigenvector. Solving for cyclic vectors we find

$$(A - \lambda I)V_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = V_1$$

$$(A - \lambda I)V_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = V_2$$

Then $P^{-1}AP = J$ where

$$P = (V_1, V_2, V_2) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \qquad J = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

Then the general solution is for some vector c,

$$e^{tA}c = Pe^{tJ}P^{-1}c = e^{-2t} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} c = e^{-2t} \begin{pmatrix} 1 + \frac{t^2}{2} & -\frac{t^2}{2} & t \\ \frac{t^2}{2} & 1 - \frac{t^2}{2} & t \\ t & -t & 1 \end{pmatrix} c$$

The exponential term kills the polynomial, so the solution tends to zero as $t \to \infty$.

2. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT: *The set of real 3×3 matrices A whose eigenvalues don't have algebraic multiplicity three are generic in the set of real 3×3 matrices.*

   TRUE. The set $\mathcal{A}$ of matrices with eigenvalues of multiplicity at most two contains the set $\mathcal{E}$ of real 3×3 matrices with distinct eigenvalues, which is open and dense in $L(\mathbf{R}^3)$, hence $\mathcal{A}$ is generic. To see that $\mathcal{E}$ is open, observe that the roots of the characteristic polynomial are isolated, so that the characteristic polynomial of nearby matrices must alo have isolated zeros, thus the eigenvalues of nearby matrices are also distinct. A matrix E not in $\mathcal{E}$ must have all real eigenvalues because if two eigenvalues are complex conjugates, the third must be real and the eigenvalues are all distinct. Let $P^{-1}EP = J$ be the Jordan form of E. Then it may be approximated by matrices $E_n = P(J + \operatorname{diag}(\frac{1}{n}, \frac{2}{n}, \frac{3}{n}))P^{-1}$ is a sequence of matrices in $\mathcal{E}$ that approximates E. Thue $\mathcal{E}$ is dense.

   (b) STATEMENT: $e^{t(A+B)} = e^{tA}e^{tB}$ *where* $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

   TRUE. The condition for $e^{t(A+B)} = e^{tA}e^{tB}$ to hold is that A and B commute. Indeed they do

$$AB = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = BA.$$

   (c) STATEMENT: *Let $\phi(t, y)$ be the solution of the initial value problem $x' = \sin(x)$ and $x(0) = y$. Then $v(t, y) = \dfrac{\partial \phi}{\partial y}(t, y)$ satisfies $v' = \cos(v)$ and $v(0) = 1$.*

   FALSE. The initial value problem satisfied by v is $v' = \cos(x(t))v$ and $v(0) = 1$ where $x(t)$ satisfies $x' = \sin(x)$ and $x(0) = y$.

3. (a) *Let* $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$. *Find* e^{tA}.

   Solving for eigenvalues, we find

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 6-\lambda \end{vmatrix} = (1-\lambda)(6-\lambda) - 2 \cdot 3 = \lambda^2 - 7\lambda = \lambda(\lambda - 7)$$

   so $\lambda_1 = 0$ and $\lambda_2 = 7$. The eigenvectors are

$$0 = (A - \lambda_1 I)V_1 = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \qquad 0 = (A - \lambda_2 I)V_1 = \begin{pmatrix} -6 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

   If we set

$$P = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}, \qquad P^{-1} = \frac{1}{7}\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

Then $P^{-1}AP = D = \operatorname{diag}\{0, 7\}$ and the exponential is

$$e^{tA} = Pe^{tD}P^{-1} = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{7t} \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 6 + e^{7t} & -2 + 2e^{7t} \\ -3 + 3e^{7t} & 1 + 6e^{7t} \end{pmatrix}$$

(b) *Determine whether the solution of the equation exists for all time and explain why.* [You don't necessarily have solve the equation.]

$$\frac{dX}{dt} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} X + \begin{pmatrix} t^3 \\ e^{t^2} \end{pmatrix}, \qquad X(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

The variation of parameters formula tell us that

$$X(t) = e^{tA}\left(x_0 + \int_0^t e^{-sA} b(s)\, ds\right)$$

$$= \frac{1}{7}\begin{pmatrix} 6 + e^{7t} & -2 + 2e^{7t} \\ -3 + 3e^{7t} & 1 + 6e^{7t} \end{pmatrix}\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \frac{1}{7}\int_0^t \begin{pmatrix} 6 + e^{-7s} & -2 + 2e^{-7s} \\ -3 + 3e^{-7s} & 1 + 6e^{-7s} \end{pmatrix}\begin{pmatrix} s^3 \\ e^{s^2} \end{pmatrix} ds\right)$$

The formula folds for all $t \in \mathbf{R}$, thus the solution exists for all time.

4. *Suppose that $f : \mathbf{R}^n \to \mathbf{R}^n$ is continuous and for some $0 < L$ we have*

$$|f(x) - f(y)| \le L|x - y| \qquad \text{for all } x, y \in \mathbf{R}^n. \tag{1}$$

(a) *Let $x_0 \in \mathbf{R}^n$. Define a sequence of functions $x_j : [0, a] \to \mathbf{R}^n$ by $x_0(t) = x_0$ and $x_{j+1}(t) = J[x_j](t)$ where*

$$J[x](t) = x_0 + \int_0^t f(x(s))\, ds.$$

Let $a = \frac{1}{2L}$. Show for all $n \ge 0$ that $x_n(t)$ is continuous and

$$|x_{n+1}(t) - x_n(t)| \le \frac{|f(x_0)|}{2^{n+1}L} \qquad \text{for all } 0 \le t \le a. \tag{2}$$

This claim is established by induction on j. For the base case $j = 0$, $x_0(t) = x_0$ is constant which is continuous and

$$x_{j+1}(t) = x_1(t) = x_0 + \int_0^t f(x_0(s))\, ds = x_0 + \int_0^t f(x_0)\, ds = x_0 + tf(x_0).$$

Thus $x_1(t)$ is linear so is a continuous function. Also for $0 \le t \le a$,

$$|x_1(t) - x_0(t)| = |x_0 + tf(x_0) - x_0| = t|f(x_0)| \le a|f(x_0)| = \frac{|f(x_0)|}{2L}$$

which is (2) for $j = 0$.
For the induction case, we assume that $x_j(t)$ is continuous and that (2) holds. By the recursion

$$x_{j+1}(t) = x_0 + \int_0^t f(x_j(s))\, ds$$

is the indefinite integral of a continuous composite of continuous functions $f(x_j(s))$, thus is continuous. Also, assuming (2) for $n = j$, for $0 \le t \le a$, using the Lipschitz condition (1) and (2)

$$\begin{aligned}
|x_{j+2}(t) - x_{j+1}(t)| &= \left| x_0 + \int_0^t f(x_{j+1}(s))\,ds - x_0 - \int_0^t f(x_j(s))\,ds \right| \\
&\le \left| \int_0^t f(x_{j+1}(s)) - f(x_j(s))\,ds \right| \\
&\le \int_0^t |f(x_{j+1}(s)) - f(x_j(s))|\,ds \\
&\le L \int_0^t |x_{j+1}(s) - x_j(s)|\,ds \\
&\le L \int_0^t \frac{|f(x_0)|}{2^{j+1}L}\,ds \\
&= \frac{Lt|f(x_0)|}{2^{j+1}L} \\
&\le \frac{La|f(x_0)|}{2^{j+1}L} \\
&= \frac{|f(x_0)|}{2^{j+2}L}
\end{aligned}$$

which is (2) for $j + 1$. This completes the proof of the induction step. Since the base and induction cases hold, we conclude that (2) holds for all j by induction.

(b) *From (a) we conclude that $\{x_j(t)\}$ is a sequence of continuous functions on $[0, a]$ that converges uniformly to a function $x_\infty(t)$. Assuming this, show that $x_\infty(t)$ satisfies a differential equation and boundary condition.*

Since $x_n(t) \to x_\infty(t)$ converges uniformly on $[0, a]$ and the x_n are continuous, it follows that x_∞ is continuous. Hence because of (1), the convergence $f(x_n(s)) \to f(x_\infty(s))$ is uniform on $[0, t]$. Uniform limits may be passed through an integral, thus for all $0 \le t \le a$,

$$\begin{aligned}
x_\infty(t) = \lim_{n\to\infty} x_{n+1}(t) &= \lim_{n\to\infty} \left(x_0 + \int_0^t f(x_n(s))\,ds \right) \\
&= x_0 + \int_0^t \lim_{n\to\infty} \Big(f(x_n(s)) \Big)\,ds = x_0 + \int_0^t f(x_\infty(s))\,ds
\end{aligned} \tag{3}$$

Thus $x_\infty(t)$ satisfies the integral equation (3). Since the right side of (3) is the integral of a continuous function, x_∞ is differentiable. By the Fundamental Theorem of Calculus, and evaluating at $t = 0$, the initial value problem satisfied by x_∞ is

$$\begin{aligned}
x_\infty' &= f(x_\infty), \\
x_\infty(0) &= x_0.
\end{aligned}$$

5. *Find the first four Picard iterates of the system. Predict the nth Picard iterate, and show that your prediction is correct. Show that the limit of the Picard iterates is a solution of the initial value problem.*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x+1 \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The first few Picard iterates are

$$X_0(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$X_1(t) = X_0 + \int_0^t F(X_0(s))\,ds = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t F\begin{pmatrix} 0 \\ 1 \end{pmatrix} ds = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} ds = \begin{pmatrix} t \\ 1+t \end{pmatrix}$$

$$X_2(t) = X_0 + \int_0^t F(X_1(s))\,ds = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t F\begin{pmatrix} s \\ 1+s \end{pmatrix} ds = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1+s \\ 1+s \end{pmatrix} ds = \begin{pmatrix} t+\frac{1}{2}t^2 \\ 1+t+\frac{1}{2}t^2 \end{pmatrix}$$

$$\begin{aligned} X_3(t) &= X_0 + \int_0^t F(X_2(s))\,ds = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t F\begin{pmatrix} s+\frac{1}{2}s^2 \\ 1+s+\frac{1}{2}s^2 \end{pmatrix} ds \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1+s+\frac{1}{2}s^2 \\ 1+s+\frac{1}{2}s^2 \end{pmatrix} ds = \begin{pmatrix} t+\frac{1}{2}t^2+\frac{1}{6}t^3 \\ 1+t+\frac{1}{2}t^2+\frac{1}{6}t^3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} X_4(t) &= X_0 + \int_0^t F(X_3(s))\,ds = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t F\begin{pmatrix} s+\frac{1}{2}s^2+\frac{1}{6}s^3 \\ 1+s+\frac{1}{2}s^2+\frac{1}{6}s^3 \end{pmatrix} ds \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1+s+\frac{1}{2}s^2+\frac{1}{6}s^3 \\ 1+s+\frac{1}{2}s^2+\frac{1}{6}s^3 \end{pmatrix} ds = \begin{pmatrix} t+\frac{1}{2}t^2+\frac{1}{6}t^3+\frac{1}{24}t^4 \\ 1+t+\frac{1}{2}t^2+\frac{1}{6}t^3+\frac{1}{24}t^4 \end{pmatrix} \end{aligned}$$

The terms seem to be exponential series. Thus we guess

$$X_n(t) = \begin{pmatrix} \sum_{k=1}^n \frac{t^k}{k!} \\ \\ \sum_{k=0}^n \frac{t^k}{k!} \end{pmatrix} \tag{4}$$

This has already beeb established in the base cases $n = 0, 1, 2, 3, 4$. Arguing by induction, assume that it is true for n to show it for $n+1$. Then

$$\begin{aligned} x_{n+1}(t) = X_0 + \int_0^t F(X_n(s))\,ds &= \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix} + \int_0^t F\begin{pmatrix} \sum_{k=1}^n \frac{s^k}{k!} \\ \\ \sum_{k=0}^n \frac{s^k}{k!} \end{pmatrix} ds \\ &= \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} \sum_{k=0}^n \frac{s^k}{k!} \\ \\ \sum_{k=0}^n \frac{s^k}{k!} \end{pmatrix} ds = \begin{pmatrix} \sum_{k=1}^{n+1} \frac{t^k}{k!} \\ \\ \sum_{k=0}^{n+1} \frac{t^k}{k!} \end{pmatrix}. \end{aligned}$$

Thus the induction step is verified and (4) holds for all n. The terms are exponential series so their limit is

$$x_\infty(t) = \begin{pmatrix} x(t) \\ \\ y(t) \end{pmatrix} = \lim_{n\to\infty} \begin{pmatrix} \sum_{k=1}^n \frac{t^k}{k!} \\ \\ \sum_{k=0}^n \frac{t^k}{k!} \end{pmatrix} = \begin{pmatrix} e^t - 1 \\ \\ e^t \end{pmatrix}$$

Finally, we check that X_∞ satisfies the IVP

$$\frac{d}{dt}\begin{pmatrix} x \\ \\ y \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} e^t - 1 \\ \\ e^t \end{pmatrix} = \begin{pmatrix} e^t \\ \\ e^t \end{pmatrix} = \begin{pmatrix} y \\ \\ x+1 \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ \\ y(0) \end{pmatrix} = \begin{pmatrix} e^0 - 1 \\ \\ e^0 \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix}.$$