Math 5410 § 1. **Second Midterm Exam** Name: Solutions
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1. *Find the general solution. Determine its behavior in* $\mathbf{R}^3$ *as* $t \to \infty$.
[Hint: the eigenvalue is $\lambda = 3$ *with algebraic multiplicity three.]*

$$X' = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 1 & 0 & 4 \end{pmatrix} X.$$

We solve for a chain of eigenvectors.

$$(A - \lambda I)V_1 = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(A - \lambda I)V_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = V_1$$

$$(A - \lambda I)V_3 = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = V_2$$

Then, put

$$T = (V_1 \mid V_2 \mid V_3\ ) = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

To see that $T^{-1}AT = J$ we compute

$$AT = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 1 & 0 & 4 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -3 & -1 & 3 \\ 3 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} = TJ$$

The general solution for constant vector c is

$$X(t) = e^{tA}c = e^{tTJT^{-1}}c = Te^{tJ}T^{-1}c = e^{3t}\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= e^{3t}\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{t^2}{2} & 1-t+\frac{t^2}{2} & t \\ t & -1+t & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = e^{3t}\begin{pmatrix} 1-\frac{t^2}{2} & t-\frac{t^2}{2} & -t \\ \frac{t^2}{2} & 1-t+\frac{t^2}{2} & t \\ t+\frac{t^2}{2} & \frac{t^2}{2} & 1+t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

As a reality check, we see that $X(0) = c$.

2. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT: *The set of real 3×3 invertible matrices A is generic in the set of real 3×3 matrices.*

   TRUE. We may describe the set of invertible matrices as those with nonvanishing determinant.

   $$S = \{A \in \mathcal{L}(\mathbf{R}^3) : \det(A) \neq 0\}.$$

   Generic means that the set contains an open and dense subset. We show S itself is open and dense. The function $f(A) = \det(A) : \mathcal{L}(\mathbf{R}^3) \to \mathbf{R}$ is continuous. Hence it pulls back open sets to open sets. But $S = f^{-1}(U)$ is the pullback of the open set $U = (-\infty, 0) \cup (0, \infty)$, hence S is open in $\mathcal{L}(\mathbf{R}^3)$. We claim that it is also dense. It suffices to show that if $A \notin S$ then there is a sequence $A_n \in S$ such that $A_n \to A$ as $n \to \infty$. Consider the matrices

   $$A_n = A + \frac{1}{n}I.$$

   Now $A_n \to A$ as $n \to \infty$. The eigenvalues of A_n are $\lambda + \frac{1}{n}$ where λ is an eigenvalue of A. Indeed, for the eigenvector $V \neq 0$ we have

   $$A_nV = (A + \frac{1}{n}I)V = AV + \frac{1}{n}V = \lambda V + \frac{1}{n}V = \left(\lambda + \frac{1}{n}\right)V.$$

   Except for at most three $n's$ the eigenvalues $\lambda + \frac{1}{n}$ are nonzero, hence $\det(A_n) \neq 0$ for these and so $A_n \in S$. Thus we have shown S is also dense in $\mathcal{L}(\mathbf{R}^3)$.

   (b) STATEMENT: *If $\omega_1 > 0$ and $\omega_2 > 0$ then the solution of the harmonic oscillator system $\ddot{x}_1 + \omega_1^2 x_1 = 0$, $\ddot{x}_2 + \omega_2^2 x_2 = 0$ with $x_1(0) = \dot{x}_1(0) = \dot{x}_2(0) = \dot{x}_2(0) = 1$ is periodic.*

   FALSE. This is true if and only if ω_2/ω_1 is rational. Writing as a first order system $y_i(t) = \dot{x}_i(t)$ in polar coordinates $x_i = r_i\cos(\theta_i)$ and $y_i = r_i\sin(\theta_i)$, we have $\dot{r}_i = 0$ so the radii remain constant and $\dot{\theta}_i = -\omega_i$ increase linearly. On the torus $r_1 = r_2 = \sqrt{2}$ the slope of the $(\theta_1(t), \theta_2(t))$ line has slope $m = \omega 2/\omega_1$ which does not close up in the torus if m is irrational. Hence the trajectory is not periodic.

(c) STATEMENT: *If $u(t) \ge 0$ is continuous and satisfies $u(t) \le 2 + 3\int_0^t u(s)\,ds$ for all $t \ge 0$ then $u(t) \le 2 + 3t$ for $t \ge 0$.*

FALSE. The Gronwall Inequality says that a continuous function satisfying $u(t) \le 2 + 3\int_0^t u(s)\,ds$ for all $t \ge 0$ satisfies $u(t) \le 2e^{3t}$ for all $t \ge 0$, not as in this statement. . To get a counterexample, consider $w(t) = 2e^{3t}$. Then for any $t \ge 0$ we have

$$2 + 3\int_0^t w(s)\,ds = 2 + 3\int_0^t 2e^{3s}\,ds = 2 + 2\left(e^{3t} - 1\right) = 2e^{3t} = w(t)$$

so the integral inequality holds. But for $t > 0$,

$$w(t) = 2e^{3t} = 2\left(1 + 3t + \sum_{k=2}^{\infty} \frac{3^k t^k}{k!}\right) > 2 + 6t$$

contrary to $w(t) \le 2 + 3t$.

3. *Let $A = \begin{pmatrix} 3 & 1 \\ -4 & 3 \end{pmatrix}$. Find e^{tA}. Solve the initial value problem. [You may leave the answer as an integral.]*

$$\frac{dX}{dt} = \begin{pmatrix} 3 & 1 \\ -4 & 3 \end{pmatrix} X + \begin{pmatrix} 1 + t^2 \\ \tan t \end{pmatrix}, \qquad X(0) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Finding eigenvalues

$$0 = \det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & 1 \\ -4 & 3 - \lambda \end{vmatrix} = (3 - \lambda)^2 + 4$$

which implies $\lambda = 3 \pm 2i$. Finding a complex eigenvector of $\lambda = 3 + 2i$ yields

$$(A - \lambda I)V = \begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix}\begin{pmatrix} 1 \\ 2i \end{pmatrix} = 0$$

A complex solution is

$$Z(t) = e^{\lambda t}V = e^{3t}(\cos(2t) + i\sin(2t))\begin{pmatrix} 1 \\ 2i \end{pmatrix} = e^{3t}\begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} + ie^{3t}\begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}.$$

A linear combination of the real and imaginary parts of this solution gives the solution with $X(0) = c$, the exponential

$$e^{tA}c = X(t) = e^{3t}\begin{pmatrix} \cos 2t & \frac{1}{2}\sin 2t \\ -2\sin 2t & \cos 2t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

To solve the inhomogeneous equation $X' = AX + b(t)$ we use the Variation of Parameters formula

$$\begin{aligned} X(t) &= e^{tA}\left\{c + \int_0^t e^{-sA}b(s)\,ds\right\} \\ &= e^{3t}\begin{pmatrix} \cos 2t & \frac{1}{2}\sin 2t \\ -2\sin 2t & \cos 2t \end{pmatrix}\left\{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \int_0^t e^{-3s}\begin{pmatrix} \cos 2s & -\frac{1}{2}\sin 2s \\ 2\sin 2s & \cos 2s \end{pmatrix}\begin{pmatrix} 1 + s^2 \\ \tan s \end{pmatrix} ds\right\} \end{aligned}$$

4. (a) *Let* $x_0 \in \mathbf{R}$. *Define a sequence of functions* $x_j : [0, \frac{1}{2}] \to \mathbf{R}$ *by* $x_0(t) = x_0$ *and* $x_{j+1}(t) = J[x_j](t)$ *where*

$$J[x](t) = x_0 + \int_0^t \sin(x(s))\, ds.$$

Show for all $n \geq 0$ that $x_n(t)$ is continuous and

$$|x_{n+1}(t) - x_n(t)| \leq \frac{1}{2^{n+1}} \qquad \text{for all } 0 \leq t \leq \frac{1}{2}.$$

[Hint: You may need the fact that $|\sin p - \sin q\,| \leq |p - q|$ for all $p, q \in \mathbf{R}$.]

(b) [5] From (a) we conclude that $\{x_j(t)\}$ is a sequence of continuous functions on $[0, \frac{1}{2}]$ that converges uniformly to a function $x_\infty(t)$. Assuming this, show that $x_\infty(t)$ satisfies a differential equation and boundary condition.

5. *Find the first four Picard iterates of the system. Predict the nth Picard iterate. Show that the limit of the Picard iterates is a solution of the initial value problem.*

$$\frac{d}{dt}\begin{pmatrix}x\\y\end{pmatrix} = F\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}y\\x\end{pmatrix}, \qquad \begin{pmatrix}x(0)\\y(0)\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}.$$

Writing vectors

$$z(t) = \begin{pmatrix}x(t)\\y(t)\end{pmatrix}, \qquad z_0 = \begin{pmatrix}1\\-1\end{pmatrix}$$

The Picard iteration yields a sequence of functions approximating the solution. The initial guess $z_1(t) = z_0$ and then for $n \in \mathbb{N}$ define

$$z_{n+1} = z_0 + \int_0^t F(z_n(t))\,ds$$

Thus the first four terms are

$$z_1(t) = \begin{pmatrix}1\\-1\end{pmatrix} + \int_0^t F\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) ds = \begin{pmatrix}1\\-1\end{pmatrix} + \int_0^t \begin{pmatrix}-1\\1\end{pmatrix} ds = \begin{pmatrix}1-t\\-1+t\end{pmatrix}$$

$$z_2(t) = \begin{pmatrix}1\\-1\end{pmatrix} + \int_0^t F\left(\begin{pmatrix}1-s\\-1+s\end{pmatrix}\right) ds = \begin{pmatrix}1\\-1\end{pmatrix} + \int_0^t \begin{pmatrix}-1+s\\1-s\end{pmatrix} ds = \begin{pmatrix}1-t+\frac{t^2}{2}\\-1+t-\frac{t^2}{2}\end{pmatrix}$$

$$z_3(t) = \begin{pmatrix}1\\-1\end{pmatrix} + \int_0^t F\left(\begin{pmatrix}1-s+\frac{s^2}{2}\\-1+s-\frac{s^2}{2}\end{pmatrix}\right) ds = \begin{pmatrix}1\\-1\end{pmatrix} + \int_0^t \begin{pmatrix}-1+s-\frac{s^2}{2}\\1-s+\frac{s^2}{2}\end{pmatrix} ds$$

$$= \begin{pmatrix}1-t+\frac{t^2}{2}-\frac{t^3}{3!}\\-1+t-\frac{t^2}{2}+\frac{t^3}{3!}\end{pmatrix}$$

$$z_4(t) = \begin{pmatrix}1\\-1\end{pmatrix} + \int_0^t F\left(\begin{pmatrix}1-t+\frac{t^2}{2}-\frac{t^3}{3!}\\-1+t-\frac{t^2}{2}+\frac{t^3}{3!}\end{pmatrix}\right) ds = \begin{pmatrix}1\\-1\end{pmatrix} + \int_0^t \begin{pmatrix}-1+t-\frac{t^2}{2}+\frac{t^3}{3!}\\1-t+\frac{t^2}{2}-\frac{t^3}{3!}\end{pmatrix} ds$$

$$= \begin{pmatrix}1-t+\frac{t^2}{2}-\frac{t^3}{3!}+\frac{t^4}{4!}\\-1+t-\frac{t^2}{2}+\frac{t^3}{6}-\frac{t^4}{4!}\end{pmatrix}$$

The limit as $n \to \infty$ seems to be

$$z_\infty(t) = \begin{pmatrix}\displaystyle\sum_{j=0}^\infty \frac{(-t)^j}{j!}\\ \displaystyle -\sum_{j=0}^\infty \frac{(-t)^j}{j!}\end{pmatrix} = \begin{pmatrix}e^{-t}\\-e^{-t}\end{pmatrix}$$

which satisfies the IVP

$$\frac{dz_\infty}{dt} = \frac{d}{dt}\begin{pmatrix}e^{-t}\\-e^{-t}\end{pmatrix} = \begin{pmatrix}-e^{-t}\\e^{-t}\end{pmatrix} = F\begin{pmatrix}e^{-t}\\-e^{-t}\end{pmatrix} = F(z_\infty),$$

$$z_\infty(0) = \begin{pmatrix}e^0\\-e^0\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} = z_0.$$