Math 5410 § 1. **Third Midterm Exam** Name: Solutions
Treibergs Nov. 15, 2023

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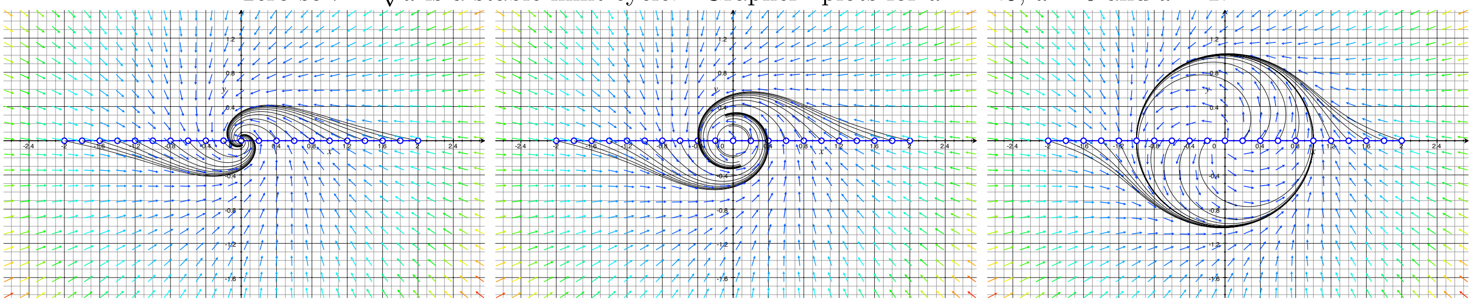
1. *Find the values of the parameter a where bifurcations occur. Describe the nature of the bifurcations. Sketch the phase plane for various a's.*

$$\dot r = r(a - r^2)$$
$$\dot\theta = 1$$

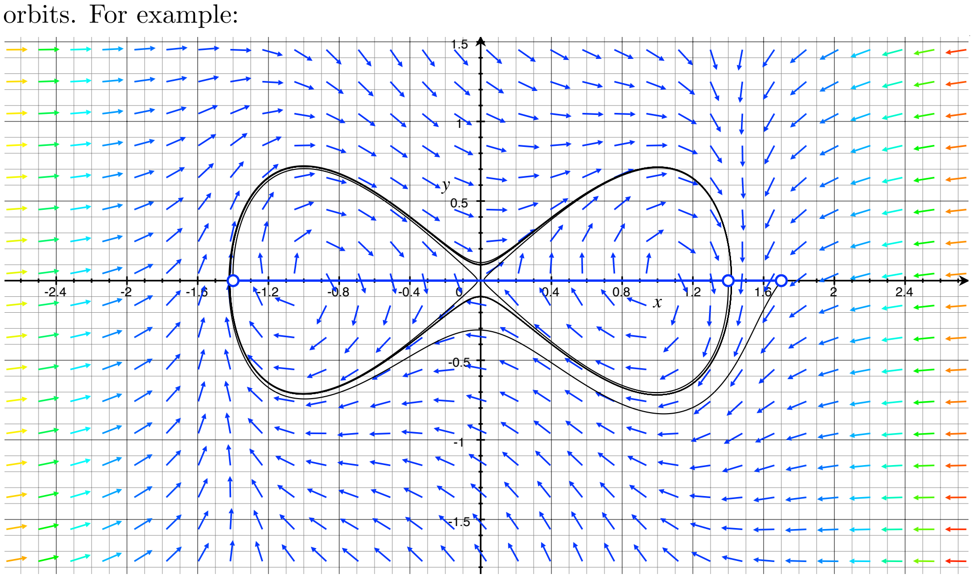
The bifurcation occurs at $a = 0$. For $a \le 0$ the origin is the only stable restpoint. For $0 < a$ the origin is still the only rest point, but it is unstable. $\dot r$ is zero at $r = \sqrt{a}$. But $\dot\theta$ is not zero so $r = \sqrt{a}$ is a stable limit cycle. "Grapher" plots for $a = -.5$, $a = 0$ and $a = 1$.



2. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) Statement: *Let $\gamma(t)$ be an entire solution of the the smooth planar system $\dot x = F(x)$. Then the omega limit set ω_γ is either a point, a nontrivial periodic orbit or the empty set.*

   False. The ω limit set is not restricted to only these three possibilities. It is possible to have ω_γ to be more complicated, consisting of the union of heteroclinic or homoclinic orbits. For example:



(b) STATEMENT: *The zero solution is asymptotically stable for the pendulum equation* $\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$ *where* b *is a positive constant.*

TRUE. The energy for the corresponding system

$$\begin{aligned}\dot{\theta} &= y \\ \dot{y} &= -\sin(\theta) - by\end{aligned}$$

is $E = \frac{1}{2}y^2 + 1 - \cos(\theta)$. It satisfies

$$\dot{E} = y\dot{y} + \sin(\theta)\dot{\theta} = y[-\sin(\theta) - by] + \sin(\theta)y = -by^2$$

Let $P = \{(\theta, y) : E(\theta, P) \le c, |\theta| \le \frac{\pi}{2}\}$ where $0 < c < 2$. On the set $Z = \{(\theta, y) : \dot{E}(\theta, y) = 0\} = \{(\theta, y) : y = 0\}$ the only invariant solution in $Z \cap P$ is $(\theta, y) = (0, 0)$. By LaSalle's Invariance Principle, $(0, 0)$ is asyptotically stable and attracts P.

(c) STATEMENT: *Every smooth planar system* $\dot{x} = F(x)$ *with* $f(0) = 0$ *is locally topologically conjugate near zero to the linear system* $\dot{y} = Ay$ *where* $A = DF_0$*.*

FALSE. This is not true unless 0 is a hyperbolic. For example, in

$$\begin{aligned}\dot{x} &= -x^3 \\ \dot{y} &= -y^3\end{aligned}$$

the origin is asymptotically stable as can be seen by using the Liapunov function $L(x, y) = x^2 + y^2$ which satisfies

$$\dot{L} = 2x\dot{x} + 2y\dot{y} = -2(x^4 + y^4) < 0$$

for $(x, y) \ne (0, 0)$. However the origin is not hyperbolic because $A = D_0F(0) = 0$ and all points of the plane are rest points for $\dot{y} = Ay$. So the nonlinear flow near the origin is not topologically conjugate to the linearized flow.

3. *Determine whether the given equilibrium point for the given system is Asymptotically Stable, is Liapunov Stable but Not Asymptotically Stable, or is Not Stable. Give a brief explanation.*

(a) $r = 1$, $\theta = 0$ for $\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = -(2 + \cos\theta)\sin\theta \end{cases}$

ASYMPTOTICALLY STABLE. This can be seen from the linearization.

$$DF(r, \theta) = \begin{pmatrix} r - 3r^2 & 0 \\ 0 & \sin^2\theta - (2 + \cos\theta)\cos\theta \end{pmatrix}, \qquad DF(1, 0) = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$$

(b) $r = 1$, $\theta = 0$ for $\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = \sin^2\left(\dfrac{\theta}{4}\right) \end{cases}$

NOT STABLE. $\dot{\theta} > 0$ for $\theta \ne 0$ thus θ is increasing away from $\theta = 0$ for $\theta > 0$ near zero but θ is heading toward $\theta = 0$ for $\theta < 0$ near zero. In fact, this system is an example for which $(1, 0)$ is attractive but not Liapunov stable.

(c) $X = (1,1)$ for $\begin{cases} \dot{x} = x(3 - x - 2y) \\ \dot{y} = y(3 - 2x - y) \end{cases}$

NOT STABLE. Linearizing

$$DF(x,y) = \begin{pmatrix} 3 - 2x - 2y & -2x \\ & \\ -2y & 3 - 2x - 2y \end{pmatrix}, \qquad DF(1,1) = \begin{pmatrix} -1 & -2 \\ & \\ -2 & -1 \end{pmatrix}$$

has determinant -4 so $(1,1)$ is a saddle and not stable.

(d) $(x,y) = (0,0)$ for $\begin{cases} \dot{x} = -x - y \\ \dot{y} = -x + y \end{cases}$

ASYMPTOTICALLY STABLE. This linear system is already in canonical form, so its eigenvalue is $\lambda = -1 \pm i$ which says that the origin is a spiral sink.

4. *Determine whether the given system admits a nontrivial limit cycle. Give a brief explanation.*

(a) $\begin{cases} \dot{x} = -2x^3 - xy^2 \\ \dot{y} = -x^2 y - 2y^3 \end{cases}$

THERE ARE NO LIMIT CYCLES. This is a gradient system

$$\begin{aligned} \dot{x} &= -\phi_x(x,y) \\ \dot{y} &= -\phi_y(x,y) \end{aligned}$$

with

$$\phi(x,y) = \frac{1}{2}\left(x^4 + x^2y^2 + y^4\right).$$

The origin is the only critical point and a global minimum of $\phi(x,y)$. Hence the trajectories all tend to the origin and are not periodic.

(b) $\begin{cases} \dot{x} = x^2 y + 2y^3 \\ \dot{y} = -2x^3 - xy^2 \end{cases}$

THERE ARE LIMIT CYCLES. This is a Hamiltonian system

$$\begin{aligned} \dot{x} &= H_y(x,y) \\ \dot{y} &= -H_x(x,y) \end{aligned}$$

with the same function

$$H(x,y) = \frac{1}{2}\left(x^4 + x^2y^2 + y^4\right).$$

Trajectories of this system are level curves of H, all of which are ovals surrounding the origin. All trajectories except the stationary point at the origin are periodic limit cycles.

(c) $\begin{cases} \dot{r} = r(2 + \sin\theta - r^2) \\ \dot{\theta} = 1 + r^2 \end{cases}$

THERE ARE LIMIT CYCLES. This follows from the Poincaré Bendixson Theorem. The vector field does not vanish except at the origin. The annulus $A = \{(r,\theta) : 1 \le r \le 2\}$ is a trapping annulus. For $r = 1$,

$$\dot{r} = r(2 + \sin\theta - r^2) \ge 1 \cdot (2 - 1 - 1^2) = 0$$

so flow is in the increasing r direction into A. For $r = 3$,

$$\dot{r} = r(2 + \sin\theta - r^2) \le 3 \cdot (2 + 1 - 2^2) = -1$$

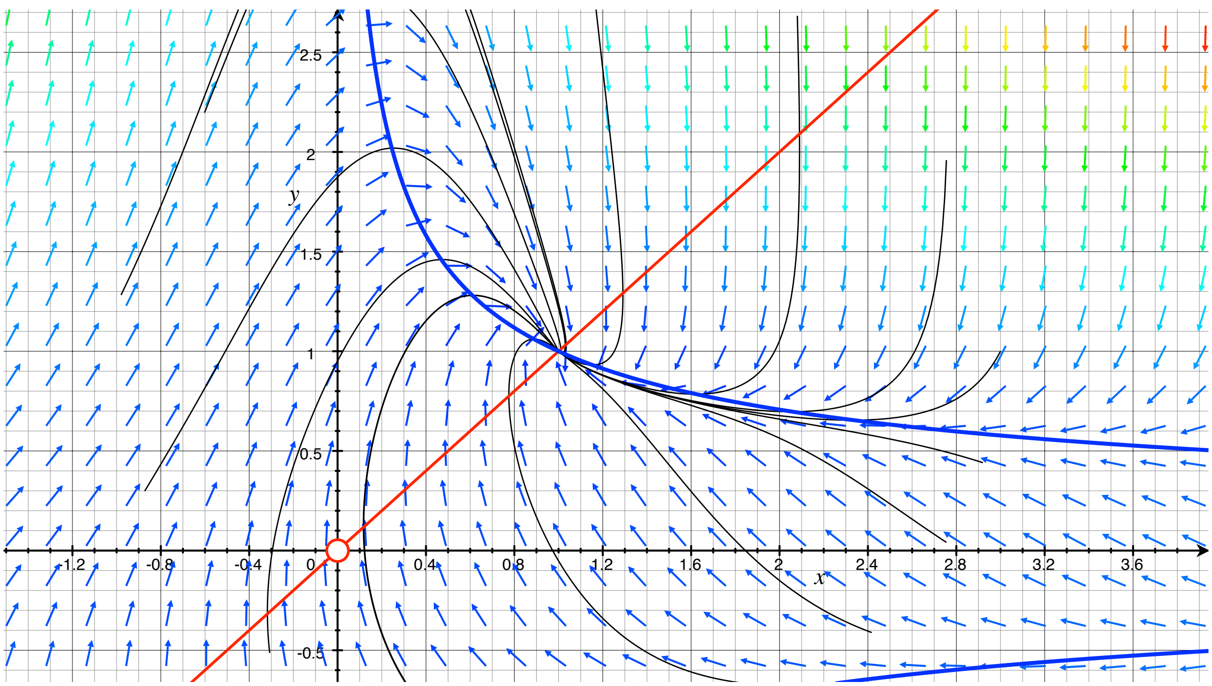
so flow is in the decreasing r direction into A as well. Since there are no rest points in A, there must be a limit cycle.

5. *The only equilibrium points is* $(1, 1)$.

$$\dot{x} = -x + y$$
$$\dot{y} = 2 - 2xy^2$$

Draw the x *and* y *nullclines and use this information to sketch the global phase portrait of this nonlinear system. At the equilibrium point* $(1, 1)$*, give a detailed description of the behavior of the linearized system.*

The $\dot{x} = 0$ nullcline is the curve $-x + y = 0$ which is colored ewd in the figure. The flow is vertical through this line. We also have $\dot{x} > 0$ if the point is above the nullcline and $\dot{x} < 0$ if it is below the nullcline. The other nullcline for $\dot{y} = 0$ is the curve given by $2 - 2xy^2 = 0$ or $x = y^{-2}$ which is colored blue in the figure. Also $\dot{y} > 0$ if the point is to the left of this curve and $\dot{y} > 0$ for points to the right. Thus in the sector above the red and blue curves, the flow goes SE with $\dot{x} < 0$ and $\dot{y} < 0$. In the sector to the right the flow is SW, Between the blue curves and to the right of the red curve the flow is NW and to the left of both the blu and red curves the flow is NE. There is only one rest point at the intersection of the blue and red curves at $(1, 1)$. We can draw the trajectories of the phase portrait following the vector field.



The Jacobian is

$$J(x, y) = \begin{pmatrix} -1 & 1 \\ -2y^2 & -4xy \end{pmatrix}, \qquad A = J(1, 1) = \begin{pmatrix} -1 & 1 \\ -2 & -4 \end{pmatrix}$$

At the rest point, the trace and determinant are $T = -5$ and $D = 6$. Thus the rest point is a sink. We have eigenvalues

$$\lambda = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 6}}{2} = \frac{-5 \pm 1}{2} \in \{-2, -3\}$$

Thus the eigenvalues of the rest point are $\lambda = -2, -3$ so the rest pont is a proper stable node. The eigenvectors corresponding to $\lambda_1 = -2$, $\lambda_2 = -3$ are

$$0 = (A - \lambda_1)v_1 = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad 0 = (A - \lambda_2)v_2 = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Thus for the two eigendirections at the rest point, the slow approach is in the $\pm v_1$ direction and the fast approach is in the $\pm v_2$ direction. Ths means that except for trajectories in the eigendirections, all other trajectories that approach $(1, 1)$ will come in tangent to the $\pm v_1$ line.