Math 5410 § 1. **Second Midterm Exam** Name: Solutions
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1. *Find the general solution. Determine its behavior in* $\mathbf{R}^3$ *as* $t \to \infty$.

$$X' = \begin{pmatrix} 0 & 1 & -6 \\ -1 & 0 & 2 \\ 0 & 0 & -3 \end{pmatrix} X$$

The matrix is block upper triangular, so its eigenvalues are $\lambda = \pm i$ and $\lambda = -3$. Solving for eigenvectors for $\lambda_1 = i$ and $\lambda_3 = -3$, we have

$$0 = (A - \lambda_1 I)W_1 = \begin{pmatrix} -i & 1 & -6 \\ -1 & -i & 2 \\ 0 & 0 & -3-i \end{pmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$

$$0 = (A - \lambda_3 I)W_3 = \begin{pmatrix} 3 & 1 & -6 \\ -1 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

One complex solution is given by

$$x(t) = e^{it}W_1 = (\cos t + i \sin t) \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix} + i \begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix}.$$

The basis of the solution space are the real and imaginary parts of the complex solution and the λ_3 solution. Thus the general solution is

$$x(t) = c_1 \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + c_3 e^{-3t} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Alternatively, changing variables $x = Ty$ makes the real canonical form of A equal to $T^{-1}AT = J$.

$$T = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \qquad T^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \qquad J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

The general solution is for arbitrary $c \in \mathbf{R}^3$,

$$x(t) = e^{tA}c = e^{tTJT^{-1}}c = Te^{tJ}T^{-1}c = T\begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^{-3t} \end{pmatrix} T^{-1}c$$

$$= \begin{pmatrix} \cos t & \sin t & -2\cos t + 2e^{-3t} \\ -\sin t & \cos t & 2\sin t \\ 0 & 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

which is the same solution in a slightly different basis to make $e^{0A} = I$.

Thus in the $\lambda = \pm i$ two-plane, the orbit is circular motion and in the $\lambda = -3$ line, the trajectory decays to zero. Thus in $\mathbf{R}^3$, all trajectories approach a particle on an elliptical orbit in a two plane as $t \to \infty$.

2. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT: *The set of real* 3×3 *hyperbolic matrices* A *is generic in the set of real* 3×3 *matrices.*

   TRUE. This set $U = \{A \in L(\mathbf{R}^3) : \Re e\, \lambda_i(A) \neq 0 \text{ for } i = 1,2,3\}$ is open and dense. The functions $\Re e\, \lambda_i : L(\mathbf{R}^3) \to \mathbf{R}$ are continuous so that $U = U_1 \cap U_2 \cap U_3$ is open because $U_i = (\Re e\, \lambda_i)^{-1}(\mathbf{R}\backslash\{0\})$ are open sets. To see that U is dense, consider a matrix $A \notin U$. Then $A_n = A + \frac{1}{n}I$ is a sequence of matrices in U that approaches A. This is because $\lambda_i(A_n) = \lambda_i(A) + \frac{1}{n}$ which is off the imaginary axis for all but possibly finitely many n's.

   (b) STATEMENT: *If* $\omega_1 > 0$ *and* $\omega_2 > 0$ *then the solution of the harmonic oscillator system* $\ddot{x} + \omega_1^2 x = 0$, $\ddot{y} + \omega_2^2 y = 0$ *with* $x(0) = \dot{x}(0) = \dot{y}(0) = \dot{y}(0) = \frac{1}{\sqrt{2}}$ *is dense in the torus* $\mathbb{S}^1 \times \mathbb{S}^1$.

   FALSE. Does not hold for all choices of ω_1 and ω_2. If $\frac{\omega_2}{\omega_1}$ is rational, then the orbit is periodic and is not dense in the torus. Writing $x_1 = \rho_1 \cos\theta_1$, $y_1 = \dot{x}_1 = \rho_1 \sin\theta_1$, $x_2 = \rho_2 \cos\theta_2$ and $y_2 = \dot{x}_2 = \rho_2 \sin\theta_2$, then $\rho_1 = \rho_2 = 1$ are constant in t so the flow is on the torus and in the (θ_1, θ_2) plane, the slope of the trajectory is $\frac{\omega_2}{\omega_1}$, so it is periodic and not dense if the ratio is rational.

   (c) STATEMENT: *If* A *and* B *are real* 2×2 *matrices then* $e^{A+B} = e^A e^B$.

   FALSE. Take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$ so $A + B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $e^A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $e^B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $e^A e^B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \neq \begin{pmatrix} \cos 1 & \sin 1 \\ -\sin 1 & \cos 1 \end{pmatrix} = e^{A+B}$.

3. (a) *Let* $A = \begin{pmatrix} 7 & 1 \\ -1 & 9 \end{pmatrix}$. *Find* e^{tA}.

   Find the eigenvalues.

$$0 = \det(A - \lambda I) = \begin{vmatrix} 7-\lambda & 1 \\ -1 & 9-\lambda \end{vmatrix} = (7-\lambda)(9-\lambda) + 1 = 64 - 16\lambda + \lambda^2 = (8-\lambda)^2$$

so $\lambda = 8, 8$. $A - 8I$ has rank one so there is only one independent eigenvector. Finding an eigenvector and a cyclic vector we take

$$(A - \lambda I)V = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$(A - \lambda I)W = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus

$$T = \begin{pmatrix} V & W \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad T^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

The Jordan form is

$$J = T^{-1}AT = \begin{pmatrix} 8 & 1 \\ 0 & 8 \end{pmatrix}.$$

To see it, we check

$$AT = \begin{pmatrix} 7 & 1 \\ -1 & 9 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 1 \\ 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 8 & 1 \\ 0 & 8 \end{pmatrix} = TJ$$

Thus the exponential is

$$e^{tA} = e^{tTJT^{-1}} = Te^{tJ}T^{-1} = e^{8t} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = e^{8t} \begin{pmatrix} 1-t & t \\ -t & t+1 \end{pmatrix}$$

(b) *Solve the initial value problem. [You may leave the answer as an integral.]*

$$\frac{dX}{dt} = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} X + \begin{pmatrix} t \\ e^t \end{pmatrix}, \qquad X(0) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The solution is given by the variation of parameters formula. The matrix is in canonical form for eigenvalue $\lambda = 3 \pm i$. Thus the exponential is

$$e^{tA} = e^{3t} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

Thus the solution is given by

$$x(t) = e^{tA}\left\{c + \int_0^t e^{-sA} g(s)\, ds\right\}$$
$$= e^{3t}\begin{pmatrix}\cos t & \sin t \\ -\sin t & \cos t\end{pmatrix}\left\{\begin{pmatrix}c_1 \\ c_2\end{pmatrix} + \int_0^t e^{-3s}\begin{pmatrix}\cos s & -\sin s \\ \sin s & \cos s\end{pmatrix}\begin{pmatrix}s \\ e^s\end{pmatrix} ds\right\}$$

4. *Find the flows* ϕ_t^X *and* ϕ_t^Y. *Find an explicit topological conjugacy* $h : \mathbf{R}^2 \to \mathbf{R}^2$ *between the flows. Check that your* h *conjugates the flows.*

$$X' = \begin{pmatrix}-2 & 0 \\ -5 & 3\end{pmatrix} X, \qquad Y' = \begin{pmatrix}-2 & 0 \\ 0 & 3\end{pmatrix} Y.$$

The second system is the diagonalization of the first, so the conjugacy will be given by the linear transformation that converts to canonical form. To find it, the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 3$. Thus

$$0 = (A - \lambda_1 I)V = \begin{pmatrix}0 & 0 \\ -5 & 5\end{pmatrix}\begin{pmatrix}1 \\ 1\end{pmatrix}; \qquad 0 = (A - \lambda_2 I)V = \begin{pmatrix}-5 & 0 \\ -5 & 0\end{pmatrix}\begin{pmatrix}0 \\ 1\end{pmatrix}$$

Thus the transformation $x = Ty$ converts the X system to the Y system, where

$$T = \begin{pmatrix}V\ W\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 1 & 1\end{pmatrix}, \qquad T^{-1} = \begin{pmatrix}1 & 0 \\ -1 & 1\end{pmatrix}$$

The topological conjugacy is given by the transformation $h(X) = T^{-1}X = Y$. You are asked to show h is a topological conjugacy between flows, and not that the transformation T converts the X system to the Y system. For this purpose, we need to find the flows. Using the fact that $B = T^{-1}AT$ we have the flows

$$\phi_t^X(\xi) = e^{tA}\xi = e^{TBT^{-1}}\xi = Te^{tB}T^{-1}\xi$$
$$\phi_t^Y(\eta) = e^{tB}\eta$$

Checking flow conjugacy we see if flowing then mapping is the same as mapping then flowing.

$$h \circ \phi_t^X(\xi) = T^{-1}\left(Te^{tB}T^{-1}\xi\right) = e^{tB}(T^{-1}\xi) = \phi_t^Y \circ h(\xi).$$

A more concrete version may be seen by working out the flows more completely.

$$\phi_t^Y(\eta) = e^{tB}\eta = \begin{pmatrix}e^{-2t} & 0 \\ 0 & e^{3t}\end{pmatrix}\begin{pmatrix}\eta_1 \\ \eta_2\end{pmatrix}$$

$$\phi_t^X(\xi) = Te^{tB}T^{-1}\xi = \begin{pmatrix}1 & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix}e^{-2t} & 0 \\ 0 & e^{3t}\end{pmatrix}\begin{pmatrix}1 & 0 \\ -1 & 1\end{pmatrix}\begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix} = \begin{pmatrix}e^{-2t} & 0 \\ e^{-2t} - e^{3t} & e^{3t}\end{pmatrix}\begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix}$$

Now we check the conjugacy equation.

$$h \circ \phi_t^X(\xi) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ e^{-2t} - e^{3t} & e^{3t} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} e^{-2t} & 0 \\ -e^{3t} & e^{3t} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$
$$= \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \phi_t^Y \circ h(\xi)$$

5. *Find the first four Picard iterates. Predict the nth Picard iterate, and show that your prediction is correct. Show that the limit of the Picard iterates is a solution of the initial value problem.*

$$\frac{dx}{dt} = x - t - 1, \qquad x(0) = 2. \tag{1}$$

By the Fundamental Theorem of Calculus, the IVP $\dot{x} = f(t,x)$, $x(t_0) = x_0$ is equivalent to the integral equation

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s))\, ds$$

In our case, (1) becomes

$$x(t) = 2 + \int_0^t x(s) - s - 1\, ds = \mathcal{N}[x](t)$$

We start the scheme at any function, say $x_0(t) = 2$ and then iterate the nonlinear operator given by the right side

$$x_{n+1}(t) = \mathcal{N}[x_n](t).$$

Let us do four iterations.

$$\begin{aligned}
x_1(t) &= 2 + \int_0^t x_0(s) - s - 1\, ds &&= 2 + \int_0^t 2 - s - 1\, ds &&= 2 + t - \frac{1}{2}t^2 \\
x_2(t) &= 2 + \int_0^t x_1(s) - s - 1\, ds &&= 2 + \int_0^t 2 + s - \frac{1}{2}s^2 - s - 1\, ds &&= 2 + t - \frac{1}{3!}t^3 \\
x_3(t) &= 2 + \int_0^t x_2(s) - s - 1\, ds &&= 2 + \int_0^t 2 + s - \frac{1}{3!}s^3 - s - 1\, ds &&= 2 + t - \frac{1}{4!}t^4 \\
x_4(t) &= 2 + \int_0^t x_3(s) - s - 1\, ds &&= 2 + \int_0^t 2 + s - \frac{1}{4!}s^4 - s - 1\, ds &&= 2 + t - \frac{1}{5!}t^3
\end{aligned}$$

It appears that

$$x_n(t) = 2 + t - \frac{1}{(n+1)!}t^{n+1}. \tag{2}$$

We check by induction. The base cases $n = 0, 1, 2, 3, 4$ have already been verified. Assume for some $n \geq 4$ that (2) holds. Using the iteration scheme and the induction hypothesis (2)

$$x_{n+1}(t) = 2 + \int_0^t x_n(s) - s - 1\, ds = 2 + \int_0^t 2 + s - \frac{1}{(n+1)!}s^{n+1} - s - 1\, ds = 2 + t - \frac{1}{(n+2)!}t^{n+2}$$

which proves the induction step. By induction, (2) holds for all n.

This sequence of functions does not converge for all real numbers, but it converges on compact subsets. Thus for any $R > 0$, on the interval $[-R, R]$ the sequence converges uniformly

$$x_n(t) = 2 + t - \frac{1}{(n+1)!} t^{n+1} \to 2 + t \qquad \text{as } n \to \infty.$$

It follows that for $|t| \le R$, the limit function solves the integral equation. Taking the limit of the recursion

$$x_{n+1}(t) = 2 + \int_0^t x_n(s) - s - 1\, ds$$

we see that because the convergence is uniform the integral and limit may be interchanged

$$\begin{aligned} x(t) = \lim_{n\to\infty} x_{n+1}(t) &= \lim_{n\to\infty} \left\{ 2 + \int_0^t x_n(s) - s - 1\, ds \right\} \\ &= 2 + \int_0^t \left\{ \lim_{n\to\infty} (x_n(s) - s - 1) \right\} ds = 2 + \int_0^t x(s) - s - 1\, ds \end{aligned}$$

which is the integral equation. Indeed, $x(t)$ is the solution of the initial value problem

$$\dot{x}(t) = x - t - 1, \qquad \text{and} \qquad x(0) = 2 + 0 = 2.$$