Midterm 1 (5440)

Name and Unid: ______________________________

Carefully Read The Instructions:

Instructions: This exam will last 50 minutes and consists of 5 exercises and one bonus exercise. Provide solutions to the exercises in the space provided or if you need extra space to work, there are two pages at the end of the test, but please idicate the exercise. All solutions MUST be sufficiently justified to receive all the credit. Illegible answers will receive deductions. Calculators, books and notes are not allowed.

Advice: If you get stuck on a exercise or a question don't panic! Move on and come back to it later.

We recall that the solution of the wave equation (on the whole real line)

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \text{ for } t > 0 \text{ and } x \in \mathbb{R},$$

with initial conditions:

$$u(x,0) = \phi(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = \psi(x)$$

is given by

$$u(x,t) = \frac{1}{2}(\phi(x+c\,t) + \phi(x-c\,t)) + \frac{1}{2c}\int_{x-c\,t}^{x+c\,t} \psi(s)\mathrm{d}s.$$

Exercise 1(Classification of partial differential equations).

1) Prove that the following partial differential equation is linear:

$$\sin(x)\,\frac{\partial^2 u}{\partial t^2}(x,t) + e^t\,u(x,t) = 0.$$

2) For each of the following equations, state the order and wether if it is nonlinear, linear inhomogeneous or linear homogenous (we don't ask any justification only for this question):

$$a) \qquad \frac{\partial^2 u}{\partial x^2}(x,t) + \cos(u(x,t)) = 0,$$

$$b) \qquad \frac{\partial^3 u}{\partial x^3}(x,t) + 2x\,\frac{\partial u^2}{\partial t^2}(x,t) + \cos(x)\frac{\partial u}{\partial x}(x,t) + 2x\,u(x,t) = 0,$$

$$c) \qquad \frac{\partial u}{\partial x}(x,t) + \frac{\partial u}{\partial t}(x,t) + x^2 u(x,t) + \cos(x) = 0,$$

3) We consider the following linear homogeneous second order partial differential equation:

$$(x^2+y^2)\,\frac{\partial^2 u}{\partial x^2}(x,y) + 4\,\frac{\partial^2 u}{\partial x \partial y}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) + 3y^2\frac{\partial u}{\partial y}(x,y) = 0\,. \qquad (1)$$

Find the regions in the xy plane where the equation (1) is elliptic, parabolic and hyperbolic. Sketch them.

Exercise 2 (Transport equation) We consider the following transport equation

$$\frac{\partial u}{\partial x}(x,y) + \cosh(x)\frac{\partial u}{\partial y}(x,y) = 0 \quad \text{for } (x,y) \in \mathbb{R}^2. \tag{2}$$

1) Find the solutions of the transport equation (2).

2) Find the solution associated to the initial condition: $u(0,y) = y^3$.

Exercise 3: (Wave equation on the whole real line).

We consider the following wave equation (with a speed $c = 1$):

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \text{ for } t > 0 \text{ and } x \in \mathbb{R}, \tag{3}$$

with initial conditions:

$$u(x,0) = \phi(x) = \begin{cases} 1 - |x| & \text{if } |x| < 1, \\ \\ 0 & \text{if } |x| \geq 1. \end{cases} \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

1) Prove that if $|x| \geq 1 + t$ then $u(x,t) = 0$. Give a physical interpretation of this condition.

2) Sketch the solutions for $t = 0$, $t = 1$ and $t = 2$.

3) Compute the energy:

$$E(t) = \frac{1}{2} \int_{\mathbb{R}} \left(\frac{\partial u}{\partial t}(x,t) \right)^2 + \left(\frac{\partial u}{\partial x}(x,t) \right)^2 \mathrm{d}x.$$

Exercise 4 (Wave equation on a Half-line)

We consider the following wave equation:

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \;\text{ for } t > 0 \text{ and } x > 0, \tag{4}$$

with the initial conditions:

$$u(x,0) = e^x \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 1\,.$$

and the homogenous Dirichlet boundary condition at $x = 0$:

$$u(0,t) = 0\,.$$

1) Using the method of reflection compute the solutions for both cases $x > c\,t$ and $0 < x < c\,t$.

2) Sketch the domain of dependence in the two following situations: $x > c\,t$ and $0 < x < c\,t$. By making a link with the first question, explain why there is a reflection phenomena in the second case.

Exercice 5 (Well-posedness of a first order linear ordinary differential equation).
We consider the following first order linear ordinary differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) + u(t) = f(t) \quad \text{and} \quad u(0) = a \in \mathbb{R} \tag{5}$$

with f a continuous function on $\mathbb{R}$.

We recall that this ordinary differential equation admits a unique solution given by the Duhamel's formula:

$$u(t) = e^{-t}\, a + \int_0^t e^{-(t-s)} f(s) \mathrm{d}s \tag{6}$$

For $T > 0$, we denote by $\|\cdot\|_T$ the uniform norm on the continuous functions on a an interval $[0, T]$:

$$\|w\|_T = \max_{0 \le t \le T} |w(t)|.$$

1) We denote by (a_1, f_1) and (a_2, f_2) the datas associated respectively to the two solutions u_1 and u_2 of the ordinary differential equation (5) and call a, f and u the quantities $a = a_2 - a_1$, $f = f_2 - f_1$ and $u = u_2 - u_1$. Prove (by showing all the steps) that for any $T > 0$,

$$\|u\|_T \le |a| + T\, \|f\|_T\, . \tag{7}$$

Deduce from the inequality (7), the stability of the ordinary differential equation (7) on the interval $[0, T]$. (Use the δ, ε definition of stability).

2) (Bonus) Prove the Duhamel's formula (6). *(Hint: Solve the ordinary differential equation satisfy by: $v(t) = e^t u(t)$ for a function u solution of (5) and conclude.)*