Quiz 1

Introduction to partial differential equations (5440)

Name and Unid: ____

1. We consider the following transport equation

$$\frac{\partial u}{\partial x}(x,y) + 4x^3y\,\frac{\partial u}{\partial y}(x,y) = 0 \quad \text{for } (x,y) \in \mathbb{R}^2. \tag{1}$$

a) Find the caracteristic curves associated to the transport equation (1) and sketch them.

b) Find the solutions of the transport equation (1).

c) Find the solution associated to the initial condition: $u(0,y) = y^4$.

2. We consider the following linear homogeneous second order partial differential equation:

$$x^2 \frac{\partial^2 u}{\partial x^2}(x,y) + 2xy \frac{\partial^2 u}{\partial x \partial y}(x,y) + (y^2 + y) \frac{\partial^2 u}{\partial y^2}(x,y) + 3y \frac{\partial u}{\partial x}(x,y) = 0\,. \tag{2}$$

a) Find the regions in the xy plane where the equation (2) is elliptic, parabolic and hyperbolic. Sketch them.

3. We consider the following modified wave equation which takes into account a transverse elastic force:

$$\frac{\partial^2 u}{\partial t^2}(x,t) - c^2 \frac{\partial^2 u}{\partial x^2}(x,t) + k\, u(x,t) = 0 \text{ for } t \geq 0 \text{ and } x \in \mathbb{R}, \tag{3}$$

where c and k are fixed positive parameters.

a) By making the assumption that the solution u of the equation (3) is compactly supported (in other words that exists a positive real number R such that if $|x| > R$, $u(x,t) = 0$), prove that the quantity:

$$E(t) = \frac{1}{2} \int_{\mathbb{R}} \left(\frac{\partial u}{\partial t}(x,t) \right)^2 + c^2 \left(\frac{\partial u}{\partial x}(x,t) \right)^2 + k\, u(x,t)^2 \, \mathrm{d}x \tag{4}$$

is a constant function (with respect to the time t), which means that it constitutes an energy of the equation (3).

b) (bonus) Suppose that u_1 and u_2 are two compactly supported solutions of the wave equation (3) with the same (regular enough) initial conditions:

$$u_1(x,0) = u_2(x,0) = \phi(x) \text{ and } \frac{\partial u_1}{\partial t}(x,0) = \frac{\partial u_2}{\partial t}(x,0) = \psi(x),$$

then by using the question a) prove that $u_1 = u_2$.