Math 5440 § 1. **Second Midterm Exam** Name: Practice
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1. *Consider the initial value problem for the diffusion equation on the line. The existence and uniqueness theorem assumes that the initial concentration $\varphi(x)$ is a bounded, piecewise continuous function. Show that a solution may or may not exist if the initial concentration is unbounded by considering three different initial conditions. In each case, find the solution or argue that it does not exist.*

$$\begin{aligned} &\text{(PDE)} \qquad & u_t &= k\, u_{xx}, \qquad &&\text{for } -\infty < x < \infty \text{ and } 0 < t; \\ &\text{(IC)} \qquad & u(x,0) &= \varphi(x), \qquad &&\text{for } -\infty < x < \infty; \end{aligned}$$

(a) $\varphi(x) = e^x$.

(b) $\varphi(x) = e^{x^2}$.

(c) $\varphi(x) = e^{x^3}$.

The solution is given by convolution with the heat kernel

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} \varphi(y)\, dy.$$

In case (a.),

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} e^y\, dy$$

Completing the square,

$$\begin{aligned} -\frac{(y-x)^2}{4kt} + y &= \frac{-(y-x)^2 + 4kty}{4kt} \\ &= \frac{-y^2 + 2(x+2kt)y - x^2}{4kt} \\ &= \frac{-y^2 + 2(x+2kt)y - (x+2kt)^2 + 4ktx + 4k^2t^2}{4kt} \\ &= \frac{-(y-x-2kt)^2}{4kt} + x + kt. \end{aligned}$$

we get

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(y-x-2kt)^2}{4kt} + x + kt}\, dy = e^{x+kt},$$

where we have used the substitution $z = y - x - 2kt$ and the total heat is one

$$\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(y-x-2kt)^2}{4kt}}\, dy = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-z^2}{4kt}}\, dz = 1.$$

One checks that the answer satisfies the IVP and exists for all (x,t). We could have guessed this solution.

In case (b.),

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} e^{y^2}\, dy$$

Completing the square again,

$$\begin{aligned}
-\frac{(y-x)^2}{4kt} + y^2 &= \frac{-(y-x)^2 + 4kty^2}{4kt} \\
&= \frac{-(1-4kt)y^2 + 2xy - x^2}{4kt} \\
&= -\frac{1-4kt}{4kt}\left(y^2 - \frac{2xy}{1-4kt} + \frac{x^2}{(1-4kt)^2}\right) + \frac{x^2}{4kt(1-4kt)} - \frac{x^2}{4kt} \\
&= -\frac{1-4kt}{4kt}\left(y - \frac{x}{1-4kt}\right)^2 + \frac{x^2}{1-4kt}.
\end{aligned}$$

we get

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{1-4kt}{4kt}\left(y - \frac{x}{1-4kt}\right)^2 + \frac{x^2}{1-4kt}}\, dy = \frac{e^{\frac{x^2}{1-4kt}}}{\sqrt{1-4kt}},$$

where we have used the substitution $z = y - \frac{x}{1-4kt}$ and the formula

$$\int_{-\infty}^{\infty} e^{-cz^2}\, dz = \sqrt{\frac{\pi}{c}}.$$

One checks that the answer satisfies the IVP and exists for all $0 < t < \frac{1}{4k}$.

In case (c.),

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} e^{y^3}\, dy \tag{1}$$

This time, for any $t > 0$,

$$-\frac{(y-x)^2}{4kt} + y^3 \to \infty$$

as $y \to \infty$ so the improper integral (1) is not even defined.

In conclusion, the integral expression may or may not be defined, depending on the growth of the function $\varphi(x)$. In case (a), the heat kernel dominates $\varphi(x)$ so the improper integral exists. Even though the initial temperature is infinitely high, it is takes infinitely long for all of space to get infinitely hot. In case (b), there as much more heat at infinity, so that as it flows in it builds up to infinite temperature in finite time: the solution increases to infinity at all x as $t \nearrow \frac{1}{4k}$. In case (c), there is so much temperature initially, that it propagates at infinite speed and all of space is infinitely hot for all $t > 0$.

2. *This problem concerns why the convolution integral provides the existence of a smooth bounded solution to the heat equation on the line for bounded, piecewise continuous initial data. It included to teach about the solution and is harder than what may be expected on an exam. Consider the initial value problem for the diffusion equation on the line. If the initial temperature $\varphi(x)$ is a bounded, piecewise continuous function, show that $u(x,t)$ given as the convolution of $\varphi(x)$ with the heat kernel results in a $\mathcal{C}^\infty$ bounded solution of the heat equation $u_t = ku_{xx}$ for $-\infty < x < \infty$ and $0 < t$, where*

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\varphi(y)\,dy, \qquad \text{where} \qquad S(z,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{z^2}{4kt}}.$$

To say $\varphi(x)$ is bounded is to say there is a constant $M < \infty$ such that

$$|\varphi(x)| \le M, \qquad \text{for all } -\infty < x < \infty.$$

$\mathcal{C}^\infty$ means that $u(x,t)$ has continuous (x,t) derivatives of any order if$-\infty < x < \infty$ and $0 < t$. φ is piecewise continuous if for any $-\infty < a \le b < \infty$, $\varphi(x,t)$ is continuous at all but finitely many points in the interval $[a,b]$. Such functions are integrable on $[a,b]$. The improper integral $u(x,t)$ converges uniformly because the integrand is dominated by the integrable function of y.

$$|S(y-x,t)\varphi(y)| \le S(y-x,t)M \qquad \text{for all } -\infty x, y < \infty \text{ and } 0 < t.$$

This inequality implies that the solution is bounded because for all $-\infty < x, y < \infty$ and $0 < t$,

$$|u(x,t)| \le \int_{-\infty}^{\infty} |S(x-y,t)\varphi(y)|\,dy \le M \int_{-\infty}^{\infty} S(x-y,t)\,dy = M.$$

Using the differentiation theorem in Appendix 3,

Theorem 1. *Let $S(z,t)$ be continuously differentiable and $\varphi(y)$ be piecewise continuous such that the integrals*

$$\int_{-\infty}^{\infty} S(x-y,t)\varphi(y)\,dy, \qquad \int_{-\infty}^{\infty} \frac{\partial S}{\partial t}(x-y,t)\varphi(y)\,dy, \qquad \int_{-\infty}^{\infty} \frac{\partial S}{\partial z}(x-y,t)\varphi(y)\,dy$$

converge uniformly as improper integrals for $t_1 < t < t_2$, then

$$\frac{\partial}{\partial x}\int_{-\infty}^{\infty} S(x-y,t)\varphi(y)\,dy = \int_{-\infty}^{\infty} \frac{\partial S}{\partial z}(x-y,t)\varphi(y)\,dy,$$
$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} S(x-y,t)\varphi(y)\,dy = \int_{-\infty}^{\infty} \frac{\partial S}{\partial t}(x-y,t)\varphi(y)\,dy.$$

One checks that in if $0 < t_1 < t_2 < \infty$, the heat kernel and its derivatives decay sufficiently to force the uniform convergence of the integrals. The derivatives

$$\frac{\partial S}{\partial z} = -\frac{z}{4\pi^{1/2}k^{3/2}t^{3/2}} e^{-\frac{z^2}{4kt}}$$
$$\frac{\partial S}{\partial t} = \frac{1}{4\pi^{1/2}k^{1/2}} e^{-\frac{z^2}{4kt}} \left(\frac{z^2}{2kt^{5/2}} - \frac{1}{t^{3/2}} \right)$$

These are bounded by integrable kernel

$$\left|\frac{\partial S}{\partial t}\right| + \left|\frac{\partial S}{\partial z}\right| \le \frac{c(1+t+z^2)}{t^{5/2}} e^{-\frac{z^2}{4kt}}$$

for some constant c depending on k and t_1. It follows that the improper integrals converge uniformly for t in (t_1, t_2). The theorem implies that u is differentiable, hence continuous there.

The kernel may be further differentiated with respect to either t or z which will generate expressions with higher z or lower t powers. In each case the exponential decay swamps the polynomial growth resulting in integrable kernel and uniformly convergent integrals. Continued application of the theorem tells us that all higher derivatives of $\dfrac{\partial^{j+k} u}{\partial^j t\, \partial^k z}$ exist and are continuous for any x and any $t > 0$.

Since $LS = S_t(x-y,t) - kS_{xx}(x-y,t) = 0$ satisfies the heat equation for every x, y and $t > 0$, it follows that the heat operator may be moved inside the integral to show that u satisfies the heat equation $Lu = 0$ as well.

[A discussion of this problem may be found in Ch. 3.5 of the text.]

3. *This problem concerns whether the convolution solution to the heat equation on the line for bounded, piecewise continuous initial data actually takes on its boundary values. It included to teach about the solution and is also harder than what may be expected on an exam. Consider the initial value problem for the diffusion equation on the line. If the initial temperature $\varphi(x)$ is a bounded, piecewise continuous function, show that $u(x,t)$ given as the convolution of $\varphi(x)$ with the heat kernel*

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\varphi(y)\,dy, \qquad \text{where} \qquad S(z,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{z^2}{4kt}}.$$

results in a extension $U(x,t)$ to the closed halfplane $t \ge 0$ which is continuous at points of continuity of φ.

$$U(x,t) = \begin{cases} u(x,t), & \text{if } t > 0, \\ \varphi(x), & \text{if } t = 0. \end{cases}$$

We have to show that $U(x,t) \to \varphi(z)$ as $(x,t) \to (z,0)$ at points z where φ is continuous. Since $\varphi(x)$ is bounded, there is $0 < B < \infty$ so that $|\varphi(x)| \le B$ for all x. Choose a point $z \in \mathbf{R}$ where $\varphi(x)$ is continuous at z.

Since the total heat is one, we have

$$u(x,t) - \varphi(z) = \int_{-\infty}^{\infty} S(x-y,t)[\varphi(y) - \varphi(z)]\,dy$$

The idea is to split the integral into two parts, one close to z where $\varphi(y) - \varphi(z)$ is small and the other, far from x where $S(x-y,t)$ is small for small t. Choose $\varepsilon > 0$. We shall show that for $|x-z|$ and t small $|u(x,t) - \varphi(z)| < \varepsilon$, so $U(x,t)$ is continuous at $(z,0)$.

The Gaussian integrand is concentrated near zero, so there is $R = R(\varepsilon, B)$ large such that

$$\frac{1}{\sqrt{\pi}} \int_{|s| \ge R} e^{-s^2}\,ds \le \frac{\varepsilon}{4B}.$$

Changing variables $y = (y-x)/\sqrt{4kt}$ we have

$$\frac{1}{\sqrt{4k\pi t}} \int_{|x-y| \ge R\sqrt{4kt}} e^{-\frac{(x-y)^2}{4kt}}\,ds \le \frac{\varepsilon}{4B}. \tag{2}$$

By the continuity of φ, there is a $\delta = \delta(\varphi, \varepsilon, z) > 0$ so that

$$|\varphi(y) - \varphi(z)| < \frac{\varepsilon}{2}, \qquad \text{whenever } |y-z| < \delta. \tag{3}$$

Splitting the integral into $|y-z| \ge \delta$ and $|y-z| \le \delta$ we find

$$|u(x,t) - \varphi(z)| = I + II.$$

Estimating I first where $|y-z| \ge \delta$, assuming that $|x-z| < \dfrac{\delta}{2}$ and $0 < t < t_0(\varepsilon, k, \varphi, z)$ is so small that $\delta > 2R\sqrt{4kt_0}$. This gives by the triangle inequality

$$|y-x| \ge |y-z| - |x-z| > \delta - \frac{\delta}{2} \ge R\sqrt{4kt}$$

and

$$|\varphi(y) - \varphi(z)| \le |\varphi(y)| + \varphi(z)| \le 2B$$

it follows by (2) that

$$\begin{aligned} I &= \frac{1}{\sqrt{4\pi kt}} \int_{|y-z|\ge\delta} e^{-\frac{(x-y)^2}{4kt}} |\varphi(y) - \varphi(z)|\, dy \\ &\le \frac{2B}{\sqrt{4\pi kt}} \int_{|x-y|\ge R\sqrt{4kt}} e^{-\frac{(x-y)^2}{4kt}}\, dy \le \frac{\varepsilon}{2}. \end{aligned}$$

Estimating II where $|y-z| < \delta$ we have by (3)

$$\begin{aligned} II &= \frac{1}{\sqrt{4\pi kt}} \int_{|y-z|<\delta} e^{-\frac{(x-y)^2}{4kt}} |\varphi(y) - \varphi(z)|\, dy \\ &\le \frac{\varepsilon}{2\sqrt{4\pi kt}} \int_{|y-z|<\delta} e^{-\frac{(x-y)^2}{4kt}}\, dy \\ &\le \frac{\varepsilon}{2\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\, dy = \frac{\varepsilon}{2}. \end{aligned}$$

We conclude that $U(x,t)$ is continuous at $(z,0)$. Given φ, z and $\varepsilon > 0$ there is $\delta(\varphi, z, \varepsilon)$ and $t_0(\varepsilon, k, \varphi, z)$ so that if $|x-z| < \delta/2$ and $0 < t < t_0$ then $|U(x,t) - U(z,0)| < \varepsilon$.

4. *Write down the propagating operator and show that Duhamel's Principle applies to the inhomogeneous transport equation.*

(PDE)	$u_t + cu_x = f(x,t),$	for $-\infty < x < \infty$ and $0 < t$;
(IC)	$u(x,0) = \varphi(x),$	for $-\infty < x < \infty$;

The propagator solves the homogeneous equation

(PDE)	$v_t + cv_x = 0,$	for $-\infty < x < \infty$ and $0 < t$;
(IC)	$v(x,0) = \varphi(x),$	for $-\infty < x < \infty$;

The solution is

$$v(x,t) = \mathcal{P}[\varphi](x,t) = \varphi(x - ct)$$

Let us solve the inhomogeneous transport equation. The characteristic satisfies

$$\dot{t} = 1, \qquad \dot{x} = c, \qquad \frac{dx}{dt} = c$$

so the characteristic through $(x,t)=(\xi,0)$ satisfies

$$x = ct+\xi.$$

Thus the evolution of $U(t)=u(\xi+ct,t)$ along the characteristic is given by the solution of

$$\frac{dU}{dt} = cu_x + u_t = f(\xi+ct,t)$$

so that

$$u(\xi+ct,t) = U(t) = U(0) + \int_0^t f(\xi+cs,s)\,ds = \varphi(\xi) + \int_0^t f(\xi+cs,s)\,ds$$

since $U(0)=u(\xi+c\cdot 0,0)=\varphi(0)$. Thus at the point (x,t), $\xi = x-ct$ so

$$u(x,t) = \varphi(x-ct) + \int_0^t f(x-ct+cs,s)\,ds.$$

Noting that

$$\mathcal{P}[f(\bullet,s)](x,t-s) = f(x-c(t-s),s)$$

we see that the formula for the solution may be written

$$u(x,t) = \mathcal{P}[\varphi](x,t) + \int_0^t \mathcal{P}[f(\bullet,s)](x,t-s)\,ds$$

whch is Duhamel's Principle for the transport equation.

5. *Solve*

$$\begin{array}{lll}
\text{(PDE)} & u_{tt}=u_{xx}, & \text{for } 0<x<\frac{\pi}{2},\ 0<t; \\
\text{(IC)} & u(x,0)=\sin x, & \text{for } 0<x<\frac{\pi}{2}; \\
 & u_t(x,0)=0, & \\
\text{(BC)} & u(0,t)=0, & \text{for } 0<t. \\
 & u_x(\frac{\pi}{2},t)=0, &
\end{array}$$

The boundary conditions ask for odd extension of $\varphi(x)$ and $\psi(x)$ at zero and even extension at $x=\frac{\pi}{2}$. The function $\varphi_{\text{ext}}(x)=\sin x$ satisfies this already! Thus the solution is obtained from d'Alembert's formula

$$\begin{aligned}
u(x,t) &= \frac{\varphi_{\text{ext}}(x-ct)+\varphi_{\text{ext}}(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi_{\text{ext}}(y)\,dy \\
&= \frac{\sin(x-ct)+\sin(x+ct)}{2} + 0.
\end{aligned}$$

6. *Find* $u(\frac{3}{4},\frac{7}{2})$ *for solution of*

$$\begin{array}{lll}
\text{(PDE)} & u_{tt}=u_{xx}, & \text{for } 0<x<1,\ 0<t; \\
\text{(IC)} & u(x,0)=x^2(1-x)^2 & \text{for } 0<x<1; \\
 & u_t(x,0)=1, & \\
\text{(BC)} & u_x(0,t)=0, & \text{for } 0<t. \\
 & u_x(1,t)=0, &
\end{array}$$

The boundary conditions ask for even extension of $\varphi(x)$ and $\psi(x)$ at $x = 0$ and $x = 1$. The function $\varphi(x)$ has zero derivatives at the endpoints and satisfies $\varphi(x) = \varphi(1-x)$. Thus the extension is just a copy o $\varphi(x)$ in every unit interval

$$\varphi_{\text{ext}} = ((x))^2[1 - ((x))]^2$$

where $((x)) = x - [x]$ is the fractional part. The even reflection at 0 and 1 of a copnmstant is just the constant $\psi_{\text{ext}}(x) = 1$. Thus the solution is obtained from d'Alembert's formula

$$\begin{aligned} u(x,t) &= \frac{\varphi_{\text{ext}}(x-ct) + \varphi_{\text{ext}}(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(y)\,dy \\ &= \frac{\varphi_{\text{ext}}(x-t) + \varphi_{\text{ext}}(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} dy \\ &= \frac{\varphi_{\text{ext}}(x-t) + \varphi_{\text{ext}}(x+t)}{2} + t. \end{aligned}$$

It follows that

$$\begin{aligned} u\left(\frac{3}{4}, \frac{7}{2}\right) &= \frac{1}{2}\left\{\varphi_{\text{ext}}\left(-\frac{11}{4}\right) + \varphi_{\text{ext}}\left(\frac{17}{4}\right)\right\} + \frac{7}{2} \\ &= \frac{1}{2}\left\{\varphi\left(\frac{1}{4}\right) + \varphi\left(\frac{1}{4}\right)\right\} + \frac{7}{2} \\ &= \left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2 + \frac{7}{2} = \frac{905}{256}. \end{aligned}$$

7. *Solve*

(PDE)	$u_{tt} - u_{xx} = \sin \pi x,$	for $0 < x < 1$, $0 < t$;
(IC)	$u(x,0) = 0,$	
	$u_t(x,0) = 0,$	for $0 < x < 1$;
(BC)	$u(0,t) = 0,$	
	$u(1,t) = 0,$	for $0 < t$.

Dirichlet boundary conditions at both ends means that we need to do odd reflections at $x = 0$ and $x = 1$. $f(x,t) = \sin \pi x$ is already odd at $x = 0$ and $x = 1$. The solution is given by d'Alembert's Formula for inhomogenous problems and $\varphi = \psi = 0$, namely,

$$\begin{aligned}
u(x,t) &= \frac{1}{2c}\int_{s=0}^{t}\int_{y=x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds \\
&= \frac{1}{2}\int_{s=0}^{t}\int_{y=x-t+s}^{x+t-s} \sin \pi y\,dy\,ds \\
&= \frac{1}{2}\int_{s=0}^{t}\left[-\frac{1}{\pi}\cos\pi y\right]_{y=x-t+s}^{x+t-s} ds \\
&= \frac{1}{2\pi}\int_{s=0}^{t} \cos\pi(x-t+s) - \cos\pi(x+t-s)\,ds \\
&= \frac{1}{2\pi^2}\Big[\sin\pi(x-t+s) + \sin\pi(x+t-s)\Big]_{s=0}^{t} \\
&= \frac{1}{2\pi^2}\Big[\sin\pi x - \sin\pi(x-t) + \sin\pi x - \sin\pi(x+t)\Big] \\
&= \frac{1}{2\pi^2}\Big[2\sin\pi x - (\sin\pi x\cos\pi t - \cos\pi x\sin\pi t) - (\sin\pi x\cos\pi t + \cos\pi x\cos\pi t)\Big] \\
&= \frac{1}{\pi^2}\sin\pi x[1-\cos\pi t].
\end{aligned}$$

One checks that this function satisfies the problem.

8. *Reduce to two ordinary differential equations, one an eigenvalue problem and the other having one initial condition, and find the particular solutions.*

$$\begin{aligned}
&\text{(PDE)} && u_{tt} + 2u_t - 4u_{xx} + u = 0, && \text{for } 0 < x < 1,\ 0 < t; \\
&\text{(IC)} && u(x,0) = 0, && \text{for } 0 < x < \frac{\pi}{2}; \\
&\text{(BC)} && u_x(0,t) = 0, && \text{for } 0 < t. \\
& && u(1,t) = 0, &&
\end{aligned}$$

Separating variables, we make the Ansatz that $u(x,t) = X(x)T(t)$. The PDE becomes

$$XT'' + 2XT' - 4X''T + XT = 0.$$

Divide by XT and separate. There is a constant λ such that

$$\frac{4X''}{X} = \frac{T'' + 2T'}{T} + 1 = -4\lambda$$

The eigenvalue problem is to find λ and nontrivial X so that

$$X'' + \lambda X = 0, \qquad X'(0) = 0, \qquad X(1) = 0.$$

Can $\lambda = 0$? If so $X'' = 0$ so

$$X(x) = A + Bx.$$

$0 = X'(0) = B$ and $0 = X(1) = A + 0\cdot 1$ implies the zero solution so zero is not an eigenvalue.

If $\lambda > 0$ we have

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

so

$$X'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$$

The boundary condition $X'(0) = 0$ implies that $B = 0$. Then the boundary condition $X(1) = 0$ implies

$$0 = x(1) = A\cos\sqrt{\lambda}.$$

If $A \neq 0$, this is possible if

$$\sqrt{\lambda} = \frac{\pi}{2} + n\pi, \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots$$

The corresponding eigenvalues are

$$\lambda_n = \pi^2\left(\frac{1}{2} + n\right)^2, \qquad n = 0, 1, 2, 3, \ldots$$

where negatives are omitted because they give the same λ_n values. The corresponding eigenfunctions are

$$X_n(x) = \cos\left(\pi\left(\frac{1}{2} + n\right)x\right), \qquad n = 0, 1, 2, 3, \ldots$$

We can rule out negative and complex eigenvalues in the same way. Suppose $\lambda \neq 0$ is complex. The characteristic equation is $\mu^2 + \lambda = 0$ whose roots are $\pm\sqrt{-\lambda}$. The general solution is

$$X(x) = Ae^{x\sqrt{-\lambda}} + Be^{-x\sqrt{-\lambda}},$$
$$X'(x) = \sqrt{-\lambda}\left(Ae^{x\sqrt{-\lambda}} - Be^{-x\sqrt{-\lambda}}\right).$$

$0 = X'(0) = \sqrt{-\lambda}(A - B)$ implies $B = A$.

$$0 = X(1) = A\left(e^{\sqrt{-\lambda}} + e^{-\sqrt{-\lambda}}\right)$$

implies $A = 0$ or

$$0 = e^{-\sqrt{-\lambda}}\left(e^{2\sqrt{-\lambda}} + 1\right)$$

Since $e^{-\sqrt{-\lambda}} \neq 0$ for all λ we must have

$$e^{2\sqrt{-\lambda}} = -1$$

so

$$2\sqrt{-\lambda} = (1 + 2n)\pi i, \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots$$

which implies

$$-\lambda = -\left(\frac{1}{2} + n\right)^2\pi^2, \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots$$

Thus any nonzero eigenvalue is positive and we have already found all of them.

The ODE for $T(t)$ is now

$$T'' + 2T' + (1 + 4\lambda)T = 0.$$

The constant coefficient equation has characteristic polynomial

$$\mu^2 + 2\mu + 1 + 4\lambda = 0$$

whose roots are

$$\mu_\pm = \frac{-2 \pm \sqrt{4 - 4(1 + 4\lambda)}}{2} = -1 \pm 2i\sqrt{\lambda}.$$

Since $\lambda > 0$, the solution is

$$T(t) = Ae^{-t} \cos\left(2t\sqrt{\lambda}\right) + Be^{-t} \sin\left(2t\sqrt{\lambda}\right)$$

Finally, the general solution of the PDE is

$$u(x,t) = \sum_{n=0}^{\infty} \cos\left(\pi\left(\tfrac{1}{2}+n\right)x\right) e^{-t} \left\{A_n \cos\left(\pi\left(1+2n\right)t\right) + B_n \sin\left(\pi\left(1+2n\right)t\right)\right\}$$

The initial condition $u(x,0) = 0$ implies $0 = T(0) = A$ so the particular solutions are

$$X_n(x)T_n(t) = \cos\left(\pi\left(\tfrac{1}{2}+n\right)x\right) e^{-t} \sin\left(\pi\left(1+2n\right)t\right), \qquad n = 0,1,2,3,\dots$$

9. *Write down the series expansion of the general solution of the damped wave equation. Assume that r is a constant such that $\pi c/\ell < r < 2\pi c/\ell$.*

$$\begin{array}{rll} \text{(PDE)} & u_{tt} + 2ru_t = c^2 u_{xx}, & \text{for } 0 < x < \ell,\ 0 < t; \\ \text{(IC)} & u(x,0) = \varphi(x), & \\ & u_t(x,0) = \psi(x), & \text{for } 0 < x < \ell; \\ \text{(BC)} & u(0,t) = 0, & \\ & u(\ell,t) = 0, & \text{for } 0 < t. \end{array}$$

Assume the solution is a product $u(x,t) = X(x)T(t)$. Plug into the PDE

$$XT'' + 2rXT' = c^2X''T$$

so there is a conastant so that

$$\frac{T'' + 2rT'}{c^2T} = \frac{X''}{X} = -\lambda$$

The eigenvalue problem for $X(x)$ is

$$X'' + \lambda X = 0, \qquad X(0) = X(\ell) = 0.$$

λ cannot be zero because if it were,

$$X(x) = A + Bx.$$

But $0 = X(0) = A$ and $0 = X(\ell) = A + B\ell$ implies $A = B = 0$ the trivial solution.

If $\lambda \neq 0$ and complex, the characteristic polynomial is $\mu^2 + \lambda = 0$ whose roots are exponents in the solution. So the general solution is

$$X(x) = Ae^{x\sqrt{-\lambda}} + Be^{-x\sqrt{-\lambda}}.$$

$0 = x(0) = A + B$ implies $B = -A$ so

$$0 = X(\ell) = A\left(e^{\ell\sqrt{-\lambda}} - e^{-\ell\sqrt{-\lambda}}\right)$$

which, in turn, implies

$$e^{2\ell\sqrt{-\lambda}} = 1$$

so

$$2\ell\sqrt{-\lambda} = 2n\pi i, \qquad n = \dots,-2,-1,0,1,2,\dots$$

Omitting zero and doubled values, it follows that all eigenvalues are positive and

$$\lambda_n = \frac{n^2\pi^2}{\ell^2}, \qquad n = 1, 2, 3, \ldots$$

The corresponding eigenfunctions are (choosing $A = \frac{1}{2i}$ and using $e^{i\theta} = \cos\theta + i\sin\theta$)

$$\begin{aligned} X_n(x) &= \frac{1}{2i}\left(e^{\frac{n\pi x i}{\ell}} - e^{-\frac{n\pi x i}{\ell}}\right) \\ &= \frac{1}{2i}\left(\cos\left(\frac{n\pi x}{\ell}\right) + i\sin\left(\frac{n\pi x}{\ell}\right) - \cos\left(\frac{n\pi x}{\ell}\right) + i\sin\left(\frac{n\pi x}{\ell}\right)\right) \\ &= \sin\left(\frac{n\pi x}{\ell}\right), \qquad n = 1, 2, 3, \ldots \end{aligned}$$

The corresponding $T(t)$ equartion is

$$T'' + 2rT' + c^2\lambda_n T = 0.$$

Its characteristic polynomial is $\mu^2 + 2r\mu + c^2\lambda_n = 0$ whose zeros are

$$\mu_\pm = \frac{-2r \pm \sqrt{4r^2 - 4c^2\lambda_n}}{2} = -r \pm \sqrt{r^2 - c^2\lambda_n}$$

Under the hypothesis $\pi c/\ell < r < 2\pi c/\ell$, the radicand

$$r^2 - c^2\lambda_n = r^2 - \frac{\pi^2 c^2 n^2}{\ell^2}$$

is positive when $n = 1$ and negative for $n = 2, 3, 4, \ldots$. Thus for $n = 1$

$$(\mu_1)_\pm = -r \pm \sqrt{r^2 - \frac{\pi^2 c^2}{\ell^2}}$$

are negative and for $n = 2, 3, 4, \ldots$

$$(\mu_n)_\pm = -r \pm i\sqrt{\frac{\pi^2 c^2}{\ell^2} - r^2}$$

are complex with negative real part. Thus the $n = 1$ time function is

$$T_1(t) = e^{-rt}\left[A_1 \exp\left(t\sqrt{r^2 - \frac{\pi^2 c^2}{\ell^2}}\right) + B_1 \exp\left(-t\sqrt{r^2 - \frac{\pi^2 c^2}{\ell^2}}\right)\right]$$

or equivalently

$$T_1(t) = e^{-rt}\left[A_1 \cosh\left(t\sqrt{r^2 - \frac{\pi^2 c^2}{\ell^2}}\right) + B_1 \sinh\left(t\sqrt{r^2 - \frac{\pi^2 c^2}{\ell^2}}\right)\right].$$

For $n = 2, 3, 4, \ldots$

$$T_n(t) = e^{-rt}\left[A_n \cos\left(t\sqrt{\frac{\pi^2 c^2 n^2}{\ell^2} - r^2}\right) + B_n \sin\left(t\sqrt{\frac{\pi^2 c^2 n^2}{\ell^2} - r^2}\right)\right].$$

The general solution is thus

$$u(x,t) = e^{-rt}\sin\left(\frac{\pi x}{\ell}\right)\left[A_1\cosh\left(t\sqrt{r^2-\frac{\pi^2c^2}{\ell^2}}\right)+B_1\sinh\left(t\sqrt{r^2-\frac{\pi^2c^2}{\ell^2}}\right)\right]$$
$$+\sum_{n=2}^{\infty} e^{-rt}\sin\left(\frac{n\pi x}{\ell}\right)\left[A_n\cos\left(t\sqrt{\frac{\pi^2c^2n^2}{\ell^2}-r^2}\right)+B_n\sin\left(t\sqrt{\frac{\pi^2c^2n^2}{\ell^2}-r^2}\right)\right]$$

The coefficients are determined by the initial conditions

$$\varphi(x) = u(x,0) = A_1\sin\left(\frac{\pi x}{\ell}\right)+\sum_{n=2}^{\infty}A_n\sin\left(\frac{n\pi x}{\ell}\right)$$
$$\psi(x) = u_t(x,0) = \left[-rA_1+B_1\sqrt{r^2-\frac{\pi^2c^2}{\ell^2}}\right]\sin\left(\frac{\pi x}{\ell}\right)$$
$$+\sum_{n=2}^{\infty}\left[-rA_n+B_n\sqrt{\frac{\pi^2c^2n^2}{\ell^2}-r^2}\right]\sin\left(\frac{n\pi x}{\ell}\right).$$

10. *Find the eigenvalues and eigenfunctions of the fourth order eigenvalue problem on* $0 < x < \ell$.

$$X'''' = \lambda X, \qquad X(0) = X'(0) = X(\ell) = X'(\ell) = 0.$$

First let us see if we can limit eigenvalues to real numbers. Assuming that λ is complex and $X(x)$ is a complex eigenfunction, we multiply by $\bar{X}$ and integrate on $[0,\ell]$ by parts

$$\int X''''\bar{X} = \lambda\int X\bar{X}$$
$$\left[X'''\bar{X}\right]_0^\ell - \int X'''\bar{X}' =$$
$$0-0-\left[X''\bar{X}'\right]_0^\ell + \int X''\bar{X}'' =$$
$$0-0+\int X''\bar{X}'' = \lambda\int X\bar{X}$$

Since an eigenfunction X cannot be constant nor linear, both $X''\bar{X}'' = |X''|^2$ and $X\bar{X} = |X|^2$ are nonnegative and not identically zero. Hence $\lambda > 0$ is positive.

Put $\lambda = \omega^4$ where ω is the positive fourth root. The characteristic equation $\mu^4 - \omega^4 = 0$ has four roots $\mu = \pm\omega, \pm i\omega$. The general solution is thus

$$X(x) = A\cosh(\omega x) + B\sinh(\omega x) + C\cos(\omega x) + D\sin(\omega x)$$

There are four boundary equations. At $x = 0$

$$0 = X(0) = A + C$$
$$0 = X'(0) = \omega(B + D)$$

It follows that $C = -A$ and $D = -B$. Then at $x = \ell$

$$0 = X(\ell) = A(\cosh\omega\ell - \cos\omega\ell) + B(\sinh\omega\ell - \sin\omega\ell)$$
$$0 = X'(\ell) = A\omega(\sinh\omega\ell + \sin\omega\ell) + B\omega(\cosh\omega\ell - \cos\omega\ell)$$

There is a nontrivial solution pair (A_n, B_n) whenever the system is singular, namely when the determinant vanishes

$$\begin{aligned}
0 &= \begin{vmatrix} \cosh\omega\ell - \cos\omega\ell & \sinh\omega\ell - \sin\omega\ell \\ \\ \sinh\omega\ell + \sin\omega\ell & \cosh\omega\ell - \cos\omega\ell \end{vmatrix} \\
&= \cosh^2\omega\ell - 2\cosh\omega\ell\cos\omega\ell + \cos^2\omega\ell - \sinh^2\omega\ell + \sin^2\omega\ell \\
&= 2 - 2\cosh\omega\ell\cos\omega\ell
\end{aligned}$$

This happens whenever

$$1 = \cosh\omega\ell\cos\omega\ell$$

As $\cosh\omega\ell > 1$, this has one solution in every interval where cosine decreases from 1 to 0 or increases from 0 to 1, namely, for $n = 1, 2, 3, \ldots$,

$$\begin{gathered}
\ell\omega_1 \in \left(0, \frac{\pi}{2}\right), \\
\ell\omega_2 \in \left(\frac{3\pi}{2}, 2\pi\right), \ell\omega_3 \in \left(2\pi, \frac{5\pi}{2}\right), \\
\ell\omega_4 \in \left(\frac{7\pi}{2}, 4\pi\right), \ell\omega_5 \in \left(4\pi, \frac{5\pi}{2}\right), \\
\cdots \\
\ell\omega_{2n} \in \left(2\pi n - \frac{\pi}{2}, 2\pi n\right), \ell\omega_{2n+1} \in \left(2\pi n, 2\pi n + \frac{\pi}{2}\right), \\
\cdots
\end{gathered}$$

Solving for (A_n, B_n), the corresponding eigenfunctions are

$$X_n(x) = (\sinh\omega_n\ell - \sin\omega_n\ell)(\cosh\omega_n x - \cos\omega_n x) - (\cosh\omega_n\ell - \cos\omega_n\ell)(\sinh\omega_n x - \sin\omega_n x)$$

11. *Solve*

$$\begin{aligned}
&\text{(PDE)} & u_t &= u_{xx}, & &\text{for } 0 < x < \pi,\ 0 < t; \\
&\text{(IC)} & u(x,0) &= \varphi(x), & &\text{for } 0 < x < \pi; \\
&\text{(BC)} & u_x(0,t) + u(0,t) &= 0, & & \\
& & u(\pi,t) &= 0, & &\text{for } 0 < t.
\end{aligned}$$

Assume the solution is a product $u(x,t) = X(x)T(t)$. Substitute into the PDE

$$XT' = X''T.$$

Separate variables

$$\frac{T'}{T} = X''X = -\lambda.$$

We get an eigenvalue problem for $X(x)$

$$X'' + \lambda X = 0, \qquad X'(0) + X(0) = 0, \qquad X(\pi) = 0.$$

Assuming that λ is complex and X is a complex eigenfunction, we multiply by $\bar{X}$ and integrate on $[0,\pi]$ by parts and using $X'(0)=-X(0)$,

$$\begin{aligned}
-\int X''\bar{X} &= \lambda\int X\bar{X}\\
-\Big[X'\bar{X}\Big]_0^{\pi}+\int X'\bar{X}' &=\\
-X'(\pi)\bar{X}(\pi)+X'(0)\bar{X}(0)+\int X'\bar{X}' &=\\
-X(0)\bar{X}(0)+\int X'\bar{X}' &= \lambda\int X\bar{X}
\end{aligned}$$

The integrals are positive and boundary term is real so λ is real. Its sign depends on the relative sizes of the terms on the left.

Assuming that $\lambda=-\gamma^2<0$, we would have the general solution

$$\begin{aligned}
X(x) &= A\cosh(\gamma x)+B\sinh(\gamma x)\\
X'(x) &= A\gamma\sinh(\gamma x)+B\gamma\cosh(\gamma x)
\end{aligned}$$

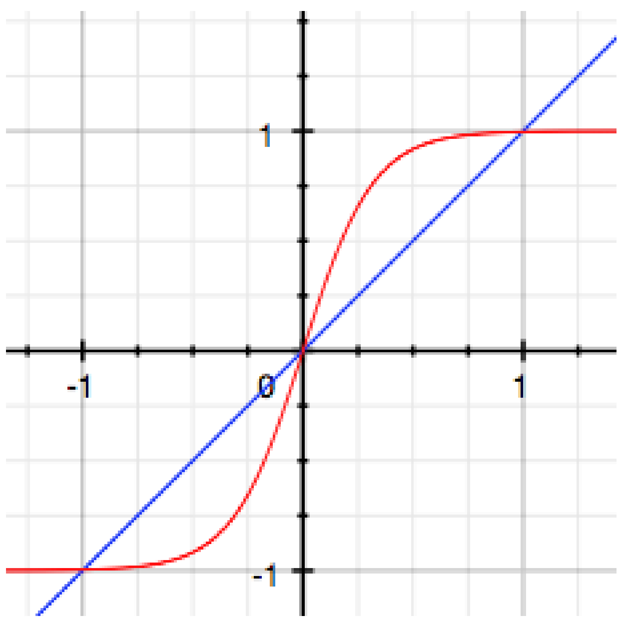
Since $0=X(\pi)$ it follows that

$$A=-B\tanh(\gamma\pi).$$

Hence

$$0=X'(0)+X(0)=\gamma B+A=B(\gamma-\tanh\gamma\pi)$$

For $B\neq 0$ there is one root where the function $y=\tanh\pi\gamma$ crosses the line $y=\gamma$ at about $\gamma_0\approx 0.9961817$. (One can easily compute the crossing point by iterating the $x_0=1$ and $x_{n+1}=\tanh(\pi x_n)$. The sequence quickly converges to γ_0 giving $\lambda_0=-\gamma^2\approx -0.992378$.



Assuming $\lambda=0$, the general solution is $X(x)=A+Bx$. $0=X(\pi)=A+B\pi$. Also $0=X'(0)+X(0)=B+A$. The two equations say $A=B=0$.

If $0<\lambda=\beta^2$ then the general solution is

$$\begin{aligned}
X(x) &= A\cos(\beta x)+B\sin(\beta x)\\
X'(x) &= -A\beta\sin(\beta x)+B\beta\cos(\beta x)
\end{aligned}$$

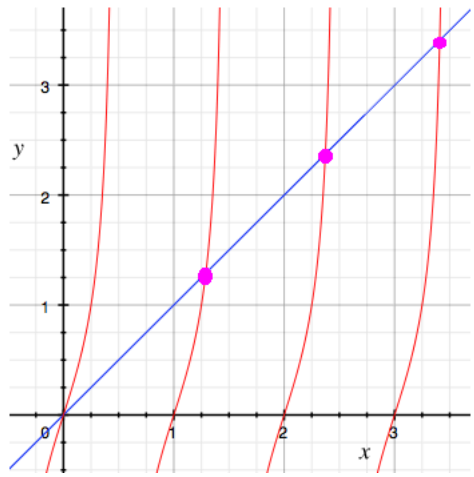
Since $0 = X(\pi)$ it follows that

$$A = -B\tan(\beta\pi).$$

Hence

$$0 = X'(0) + X(0) = \beta B + A = B(\beta - \tan\beta\pi)$$

For $B \neq 0$ there are infinitely many roots of $\beta = \tan\pi\beta$, one in each interval $\beta_n \in (n, n+1)$ for $n = 1, 2, 3, \ldots$. By iterating the inverse relation, $x_0 = n$ and $x_{k+1} = n + \text{Atan}(x_k)/\pi$ we see that $\beta_1 = 1.29011$ and $\beta_2 = 2.373053$ so $\lambda_1 \approx 1.664383$ and $\lambda_2 \approx 5.63138$.



The eigenfunctions are

$$X_0(x) = -\tanh(\gamma_0\pi)\cosh(\gamma_0 x) + \sinh(\gamma_0 x)$$
$$X_n(x) = -\tan(\beta_n\pi)\cos(\beta_n x) + \sin(\beta_n x), \qquad n = 1, 2, 3, \ldots$$

The ODE for $T(t)$ is

$$T' + \lambda_n T = 0$$

whose solution is $T_n(t) = e^{-\lambda_n t}$. It follows that the general solution is

$$u(x,t) = A_0 e^{\gamma_0^2 t}\{-\tanh(\gamma_0\pi)\cosh(\gamma_0 x) + \sinh(\gamma_0 x)\} + \sum_{n=1}^{\infty} A_n e^{-\beta_n^2 t}\{-\tan(\beta_n\pi)\cos(\beta_n x) + \sin(\beta_n x)\}$$

and the initial data determines the coefficient

$$\varphi(x) = u(x,0) = A_0\{-\tanh(\gamma_0\pi)\cosh(\gamma_0 x) + \sinh(\gamma_0 x)\} + \sum_{n=1}^{\infty} A_n\{-\tan(\beta_n\pi)\cos(\beta_n x) + \sin(\beta_n x)\}.$$

12. *Find the Fourier cosine series for $x^2/2$ on $(0,\ell)$ by assuming you can integrate the sine series for x term by term on $(0,\ell)$. Use the formula to determine the value of the infinite sum*

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

The coefficients of the sine series

$$\varphi(x) \sim \sum_{m=1}^{\infty} B_n \sin\left(\frac{\pi m x}{\ell}\right)$$

may be picked off by integrating against $\sin(n\pi x/\ell)$ over $(0,\ell)$ because the integrals of the cross terms $m \neq n$ vanish. Hence, integrating by parts with $u = x$ and $dv = \sin(n\pi x/\ell)\,dx$,

$$\begin{aligned}
\frac{B_n \ell}{2} = B_n \int_0^\ell \sin^2\left(\frac{n\pi x}{\ell}\right)\,dx &= \int_0^\ell x \sin\left(\frac{n\pi x}{\ell}\right)\,dx \\
&= -\frac{\ell}{n\pi}\left[x\cos\left(\frac{n\pi x}{\ell}\right)\right]_0^\ell + \frac{\ell}{n\pi}\int_0^\ell \cos\left(\frac{n\pi x}{\ell}\right)\,dx \\
&= -\frac{\ell^2}{n\pi}\cos(n\pi) + \frac{\ell^2}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{\ell}\right)\right]_0^\ell \\
&= -\frac{\ell^2(-1)^n}{n\pi} + 0 - 0
\end{aligned}$$

It follows that

$$x \sim \frac{2\ell}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{\pi n x}{\ell}\right)$$

Note that

$$\int_0^x \sin\left(\frac{n\pi y}{\ell}\right)\,dy = \frac{\ell}{n\pi}\left[-\cos\left(\frac{\pi n x}{\ell}\right)\right]_0^x = \frac{\ell}{n\pi}\left[1 - \cos\left(\frac{\pi m x}{\ell}\right)\right]$$

The formal term-by-term integration on $\int_0^x$ is the series

$$\frac{x^2}{2} \sim \frac{2\ell^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\left[1 - \cos\left(\frac{\pi m x}{\ell}\right)\right] = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi n x}{\ell}\right)$$

where

$$A_0 = \frac{4\ell^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}, \qquad A_n = -\frac{2\ell^2(-1)^{n+1}}{\pi^2 n^2}, \qquad n = 1, 2, 3, \ldots$$

Finally, the formula for the constant term in the cosine series gives

$$A_0 = \frac{2}{\ell}\int_0^\ell \varphi(y)\,dy = \frac{1}{\ell}\int_0^\ell y^2\,dy = \frac{\ell^2}{3}.$$

Equating the two experssions for A_0 yields

$$S = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

13. *Find the solution, including the coefficients.*

$$\begin{aligned}
&\text{(PDE)} \qquad & u_t &= u_{xx}, & &\text{for } 0 < x < 1,\ 0 < t; \\
&\text{(IC)} \qquad & u(x,0) &= \begin{cases} \frac{5x}{2}, & \text{for } 0 < x < \frac{2}{3}, \\ 3 - 2x, & \text{for } \frac{2}{3} < x < 1; \end{cases} \\
&\text{(BC)} \qquad & u(0,t) &= 0, \\
& & u(1,t) &= 1, & &\text{for } 0 < t.
\end{aligned}$$

Put $u(x,t) = x + v(x,t)$, which gives v Dirichlet boundary conditions. $v(x,t)$ satisfies

$$\begin{aligned}
&\text{(PDE)} \qquad && v_t = v_{xx}, && \text{for } 0 < x < 1,\ 0 < t; \\
&\text{(IC)} \qquad && v(x,0) = \begin{cases} \frac{3x}{2}, & \text{for } 0 < x < \frac{2}{3}, \\ 3 - 3x, & \text{for } \frac{2}{3} < x < 1; \end{cases} \\
&\text{(BC)} \qquad && v(0,t) = 0, \\
& && v(1,t) = 0, && \text{for } 0 < t.
\end{aligned}$$

For the Ansatz $v(x,t) = X(x)T(t)$, the PDE is

$$XT' = X''T$$

sp

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda.$$

The resulting eigenvalue problem is with both Dirichlet conditions

$$X'' + \lambda X = 0, \qquad X(0) = X(1) = 0.$$

Its eigenfunctions and eigenvalues are

$$X_n(x) = \sin(\pi n x), \qquad \lambda_n = \pi^2 n^2, \qquad n = 1, 2, 3, \ldots$$

Hence the time equation and its solution becomes

$$T' + \lambda_n T = 0; \qquad T_n(t) = e^{-\pi^2 n^2 t}$$

The superposition of the Ansätze becomes the general solution

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin(\pi n x) e^{-\pi^2 n^2 t}$$

Hence, the initial condition becomes

$$\varphi(x) = v(x,0) = \sum_{n=1}^{\infty} B_n \sin(\pi n x)$$

The Fourier sine coefficient has the formula

$$B_n = 2\int_0^1 \varphi(x) \sin(\pi n x)\, dx = 3\int_0^{\frac{2}{3}} x \sin \pi n x\, dx + 6\int_{\frac{2}{3}}^1 (1-x) \sin \pi n x\, dx$$

These integrals may be computed by integration by parts. In the first, take $u = 3x$ and $dv = \sin \pi n x\, dx$ so $v = -\frac{1}{\pi n} \cos(\pi n x)$

$$\begin{aligned}
3\int_0^{\frac{2}{3}} x \sin \pi n x\, dx &= -\frac{3}{\pi n}\Big[x \cos \pi n x\Big]_0^{\frac{2}{3}} + \frac{3}{\pi n}\int_0^{\frac{2}{3}} \cos \pi n x\, dx \\
&= -\frac{2}{\pi n} \cos\left(\frac{2\pi n}{3}\right) + \frac{3}{\pi^2 n^2}\Big[\sin \pi n x\Big]_0^{\frac{2}{3}} \\
&= -\frac{2}{\pi n} \cos\left(\frac{2\pi n}{3}\right) + \frac{3}{\pi^2 n^2} \sin\left(\frac{2\pi n}{3}\right)
\end{aligned}$$

In the second, take $u = 6 - 6x$ and $dv = \sin \pi n x\, dx$ so $v = -\frac{1}{\pi n}\cos(\pi n x)$

$$\int_{\frac{2}{3}}^{1} (6-6x)\sin \pi n x\, dx = -\frac{6}{\pi n}\Big[(1-x)\cos \pi n x\Big]_{\frac{2}{3}}^{1} - \frac{6}{\pi n}\int_{\frac{2}{3}}^{1} \cos \pi n x\, dx$$
$$= \frac{2}{\pi n}\cos\left(\frac{2\pi n}{3}\right) - \frac{6}{\pi^2 n^2}\Big[\sin \pi n x\Big]_{\frac{2}{3}}^{1}$$
$$= \frac{2}{\pi n}\cos\left(\frac{2\pi n}{3}\right) + \frac{3}{\pi^2 n^2}\sin\left(\frac{2\pi n}{3}\right)$$

Adding we get

$$B_n = \frac{6}{\pi^2 n^2}\sin\left(\frac{2\pi n}{3}\right) = \begin{cases} 0, & \text{if } n \equiv 0 \mod 3; \\ \dfrac{3\sqrt{3}}{\pi^2 n^2}, & \text{if } n \equiv 1 \mod 3; \\ -\dfrac{3\sqrt{3}}{\pi^2 n^2}, & \text{if } n \equiv 2 \mod 3; \end{cases}$$

Rewriting the solution one finds $u(x,t) = x + v(x,t) =$

$$x + \frac{3\sqrt{3}}{\pi^2}\left(\sin(\pi x)e^{-\pi^2 t} - \frac{\sin(2\pi x)e^{-4\pi^2 t}}{4} + \frac{\sin(4\pi x)e^{-16\pi^2 t}}{16} - \frac{\sin(5\pi x)e^{-25\pi^2 t}}{25} + \cdots\right)$$

14. *Show that the eigenfunctions corresponding to different eigenvalues of the following problem are orthogonal. (Hint: You do not need to find the eigenfunctions explicitly.)*

(O.D.E.)	$X'' + \lambda X = 0,$	for $-1 < x < 1;$
(B. C.)	$X(-1) + X'(-1) = 0;$	
	$X(1) - X'(1) = 0;$	

Suppose that X_n and X_m be eigenfunctions corresponding to different eigenvalues $\lambda_n \neq \lambda_m$. Since this eigenvalue problem with Robin boundary conditions is known to be real, orthogonality means the vanishing to the inner product integral. Using the ODE twice, integrating by parts twice and using the boundary conditions we find

$$\begin{aligned}\lambda_n(X_m, X_n) &= \int_{-1}^{1} X_m \cdot \lambda_n X_n \\ &= -\int_{-1}^{1} X_m \cdot X_n'' \\ &= \int_{-1}^{1} X_m' \cdot X_n' - [X_m(1)X_n'(1) - X_m(-1)X_n'(-1)] \\ &= -\int_{-1}^{1} X_m'' \cdot X_n + [X_m'(1)X_n(1) - X_m'(-1)X_n(-1)] \\ &\qquad - [X_m(1)X_n'(1) - X_m(-1)X_n'(-1)] \\ &= \lambda_m \int_{-1}^{1} X_m \cdot X_n + [X_m(1)X_n(1) + X_m(-1)X_n(-1)] \\ &\qquad - [X_m(1)X_n(1) + X_m(-1)X_n(-1)] \\ &= \lambda_m(X_m, X_n) + 0.\end{aligned}$$

Subtracting implies
$$0 = (\lambda_n - \lambda_m)(X_m, X_n).$$
But $\lambda_n \neq \lambda_m$ implies the eigenfunctions are orthogonal $(X_m, X_n) = 0$.

15. *(a) Let $\mathbf{R}_2[X]$ be the vector space of polynomials with real coefficients of degree at most two. Prove that:*
$$\langle P, Q\rangle = P(0)Q(0) + P(1)Q(1) + P(2)Q(2),\ \forall P, Q \in \mathbf{R}_2[X]$$
defines a real inner product on $\mathbf{R}_2[X]$.

(b) Let $\mathcal{H}$ be a real vector space equipped with a real inner product $\langle\, ,\, \rangle$ which defines a L^2 norm $\|\cdot\|$ by:
$$\|X\| = (X, X)^{\frac{1}{2}},\ \ ,\forall X \in \mathcal{H} \tag{4}$$
Prove the following identity (referred as the parallelogram identity):
$$\|X + Y\|^2 + \|X - Y\|^2 = 2(\|X\|^2 + \|Y\|^2),\ \forall X, Y \in \mathcal{H} \tag{5}$$

(c) Interpret geometrically the identity (5) in the case where $\mathcal{H} = \mathbf{R}^2$ and
$$\langle X, Y\rangle = x_1 y_1 + x_2 y_2,\ \forall X = (x_1, x_2),\ Y = (y_1, y_2) \in \mathbf{R}^2.$$
[Hint: think parallelogram.]

(a) For any $f, g, h \in \mathbf{V}$, a real vector space, the real function (f, g) is an inner product if it is (1) symmetric $(f, g) = (g, f)$, (2) bilinear $(\alpha f + \beta g, h) = \alpha(f, h) + \beta(g, h)$ (so $(f, \alpha g + \beta h) = \alpha(f, h) + \beta(f, h)$ by symmetry) for any $\alpha, \beta \in \mathbf{R}$ and (3) positive definite $(f, f) \geq 0$ and $(f, f) = 0 \implies f = 0$.

Let us check the three properties on $\langle \bullet, \bullet\rangle$. For any three polynomials P, Q and R and two numbers α, β,
$$\begin{aligned}\langle P, Q\rangle &= P(0)Q(0) + P(1)Q(1) + P(2)Q(2)\\ &= Q(0)P(0) + Q(1)P(1) + Q(2)P(2)\\ &= \langle P, Q\rangle\end{aligned}$$
so $\langle \bullet, \bullet\rangle$ is symmetric.
$$\begin{aligned}\langle \alpha P + \beta Q, R\rangle &= (\alpha P + \beta Q)(0)R(0) + (\alpha P + \beta Q)(1)R(1) + (\alpha P + \beta Q)(2)R(2)\\ &= \alpha\{P(0)R(0) + P(1)R(1) + P(2)R(2)\} + \beta\{Q(0)R(0) + Q(1)R(1) + Q(2)R(2)\}\\ &= \alpha\langle P, R\rangle + \beta\langle Q, R\rangle\end{aligned}$$
so $\langle \bullet, \bullet\rangle$ is bilinear.
$$\langle P, P\rangle = P(0)^2 + P(1)^2 + P(2)^2 \geq 0$$

Since squares are nonnegative. If $\langle P, P\rangle = 0$ then each term vanishes
$$P(0) = P(1) = P(2) = 0.$$
If the polynomial is $P(x) = \alpha + \beta x + \gamma x^2$, this says
$$\begin{aligned}P(0) &= \alpha &&= 0\\ P(1) &= \alpha + \beta + \gamma &&= 0\\ P(2) &= \alpha + 2\beta + 4\gamma &&= 0\end{aligned}$$

which implies $\alpha = \beta = \gamma = 0$ so $P(x) = 0$ is the zero polynomial. So $\langle \bullet, \bullet \rangle$ is positive definite. Thus the three conditions for inner product are verified.

(b) Let $X, Y \in \mathcal{H}$. We have using linearity in both slots and symmetry

$$\begin{aligned}
\|X+Y\|^2 + \|X-Y\|^2 &= \langle X+Y, X+Y\rangle + \langle X-Y, X-Y\rangle \\
&= \{\langle X, X+Y\rangle + \langle Y, X+Y\rangle\} + \{\langle X, X-Y\rangle - \langle Y, X-Y\rangle\} \\
&= \{\langle X, X\rangle + \langle X, Y\rangle + \langle Y, X\rangle + \langle Y, Y\rangle\} + \\
&\quad \{\langle X, X\rangle - \langle X, Y\rangle - \langle Y, X\rangle + \langle Y, Y\rangle\} \\
&= 2\langle X, X\rangle + 2\langle Y, Y\rangle \\
&= 2\|X\|^2 + 2\|Y\|^2
\end{aligned}$$

(c) In $\mathbf{R}^2$, $\|(x_1, x_2)\|^2 = \langle (x_1, x_2), (x_1, x_2)\rangle = x_1^2 + x_2^2$ is the square of the Euclidean length. (5) says that for the quadrilateral in the plane with vertices 0, X, Y and $X+Y$ the sum of squares of lengths of diagonals $\|X+Y\|^2 + \|X-Y\|^2$ equals twice the squares of lengths of generating sides $2\|X\|^2 + 2\|Y\|^2$.

16. *Consider any series of functions on any finite interval.*

    (a) Show that if it converges uniformly, then it also converges pointwise.

    (b) Show that if it converges uniformly, then it also converges in the $\mathcal{L}^2$ sense.

Take $I = [a, b]$ to be the finite interval and $\{f_n\}$ the functions. Denote the partial sum by

$$S_n(x) = \sum_{k=1}^{n} f_k(x).$$

The series is said to *converge uniformly* to a function $S(x)$ on I if for every $\varepsilon > 0$ there is a $N = N(\varepsilon) \in \mathbf{R}$ such that

$$|S_n(x) - S(x)| < \varepsilon \qquad \text{whenever } x \in I \text{ and } n \geq N.$$

(a) The series is said to *converge pointwise* to a function $S(x)$ on I if for every $\varepsilon > 0$ and every $x \in I$ there is a $N(x, \varepsilon) \in \mathbf{R}$ such that

$$|S_n(x) - S(x)| < \varepsilon \qquad \text{whenever } n \geq N.$$

To see uniform convergence implies pointwise convergence, choose $\varepsilon > 0$ and $x \in I$. Take $N(\epsilon)$ from the uniform convergence. Then $n \geq N$ implies

$$|S_n(y) - S(y)| < \varepsilon \qquad \text{whenever } y \in I \text{ and } n \geq N.$$

In particular, for $y = x$,

$$|S_n(x) - S(x)| < \varepsilon \qquad \text{whenever } n \geq N.$$

Hence $S_n(x) \to S(x)$ as $n \to \infty$ for every $x \in I$. Hence $S_n \to S$ pointwise on I.

(b) The series is said to *converge in the $\mathcal{L}^2$-sense* to a function $S(x)$ on I if for every $\varepsilon > 0$ there is a $N(\varepsilon) \in \mathbf{R}$ such that

$$\int_I |S_n(x) - S(x)|^2 \, dx < \varepsilon \qquad \text{whenever } n \geq N.$$

To see uniform convergence implies $\mathcal{L}^2$ convergence, choose $\varepsilon > 0$. Take $N = N\left(\sqrt{\frac{\epsilon}{b-a}}\right)$ from the uniform convergence. Then $n \geq N$ implies

$$|S_n(x) - S(x)| < \sqrt{\frac{\varepsilon}{b-a}} \qquad \text{whenever } x \in I \text{ and } n \geq N.$$

Integrating the square,

$$\int_I |S_n(x) - S(x)|^2 \, dx < \int_a^b \frac{\varepsilon}{b-a} \, dx = \varepsilon \qquad \text{whenever } n \geq N.$$

Hence $S_n \to S$ in the $\mathcal{L}^2$-sense on I.