Math 544 § 1. **Final Exam** Name:________________________

Treibergs Dec. 13, 1994

This is a closed book test. No books, papers, calculators.
There are [120] total points.

1.______	/15
2.______	/15
3.______	/15
4.______	/15
5.______	/15
6.______	/15
7.______	/15
8.______	/15
Total______	/120

1. (a) Using separation of variables, solve the heat equation.

$$\begin{array}{lll} \text{(P.D.E.)} & u_t = k u_{xx}, & \text{for } 0 < x < \pi,\ 0 \le t; \\ \text{(I. C.)} & u(x,0) = \sin(x) + \sin(2x) + \sin(3x), & \text{for } 0 < x < \pi. \\ \text{(B. C.)} & u(0,t) = u(\pi,t) = 0, & \text{for } 0 < t. \end{array}$$

(b) By substituting your solution directly, show that the energy is a decreasing function:

$$E(t) = \int_0^\pi u^2(s,t)\,ds.$$

2. Find the general solution of the wave equation. Then find the particular solution with the given initial data You may leave the answer in terms of integrals.

$$\begin{array}{llll} \text{(P.D.E.)} & u_{tt} - c^2 u_{xx} = 0, & & \text{for } 0 < x < \frac{\pi}{2},\ 0 \le t; \\ \text{(B. C.)} & u(0,t) = 0, & u_x(\frac{\pi}{2}, t) = 0, & \text{for } 0 < t; \\ \text{(I. C.)} & u(x,0) = 0, & u_t(x,0) = x, & \text{for } 0 < x < \frac{\pi}{2}; \end{array}$$

Hint: The eigenvalue problem

$$X'' + \lambda X = 0; \qquad X(0) = X'(\frac{\pi}{2}) = 0;$$

has the eigenvalues $\lambda_n = (2n+1)^2$, $n = 0, 1, 2, \ldots$ and corresponding eigenfunctions $X_n(x) = \sin(2n+1)x$.

3. Find all the eigenvalues and eigenfunctions for the problem

$$\begin{array}{lll} \text{(O.D.E.)} & X'' + \lambda X = 0, & \text{for } 0 < x < 1; \\ \text{(B. C.)} & X(0) + 2X'(0) = 0; & \\ & X(1) - 2X'(1) = 0; & \end{array}$$

4. Show that the eigenfunctions corresponding to different eigenvalues of the following problem are orthogonal. (Hint: You do not need to find the eigenfunctions explicitly.)

$$\begin{array}{lll} \text{(O.D.E.)} & X'' + \lambda X = 0, & \text{for } -1 < x < 1; \\ \text{(B. C.)} & X(-1) + X'(-1) = 0; & \\ & X(1) - X'(1) = 0; & \end{array}$$

5. Let D denote the closed disk $D = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 \le 1\}$. Suppose that $u(x,y) \in \mathcal{C}^2\big(D\big)$ is a solution of the equation

$$u_{xx} + u_{yy} = 2 \qquad \text{if } (x,y) \in D.$$

Prove that

$$\max_{(x,y)\in D} u(x,y) \le \max_{(x,y)\in \partial D} u(x,y).$$

Hint: try the same argument that works in the parabolic case.

6. Solve the initial–boundary value problem for the inhomogeneous wave equation

$$\begin{array}{llll} \text{(P.D.E.)} & u_{tt} - u_{xx} = \sin x, & & \text{for } 0 \le x < \infty,\ 0 \le t < \infty; \\ \text{(I. C.)} & u(x,0) = 0, & u_t(x,0) = \dfrac{1}{1+x^2}, & \text{for } 0 \le x < \infty; \\ \text{(B. C.)} & u(0,t) = 0, & & \text{for } 0 \le t < \infty; \end{array}$$

7. Solve the inhomogeneous heat equation

$$\begin{array}{lll} \text{(P.D.E.)} & u_t - k u_{xx} = x^2 t^3, & \text{for } -\infty < x < \infty,\ 0 \le t; \\ \text{(I. C.)} & u(x,0) = \operatorname{sech}(x), & \text{for } -\infty < x < \infty; \end{array}$$

You may leave your answer as an integral.

8. The transport, heat and wave equations are all evolution equations on $\mathbf{R} \times [0,\infty)$. They all take a function $f(\bullet)$ at time $t = 0$ and evolve it to $u(\bullet, t)$ at time t. (a, c, k are constant.)

$$\begin{array}{llll} \text{(T)} & u_t - a u_x = 0; & & \text{for } -\infty < x < \infty,\ 0 \le t; \\ & u(x,0) = f(x) & & \text{for } -\infty < x < \infty; \end{array}$$

$$\begin{array}{llll} \text{(H)} & v_t - k v_{xx} = 0; & & \text{for } -\infty < x < \infty,\ 0 \le t; \\ & v(x,0) = f(x) & & \text{for } -\infty < x < \infty; \end{array}$$

$$\begin{array}{llll} \text{(W)} & w_{tt} - c^2 w_{xx} = 0; & & \text{for } -\infty < x < \infty,\ 0 \le t; \\ & w(x,0) = f(x), & w_t(x,0) = 0 & \text{for } -\infty < x < \infty; \end{array}$$

Explain why each of these flows is different. For each pair of evolutions, give some property which distinguishes one from the other. Be brief. Be specific. Your answer should fit on this piece of paper. *e.g.,* One property all three have in common is that for $f \in \mathcal{C}^2_c(\mathbf{R})$ (twice continuously differentiable with compact support) there is existence of a solution for all $t \ge 0$. This is seen by the availability of formulas for solutions for all three in terms of f. Demonstrate this by writing the three formulas.