Homework for Math 5440 §1, Fall 2016

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November 22, 2016

Our text is by Walter A. Strauss, *Introduction to Partial Differential Equations* 2nd ed., Wiley, 2007. Please read the relevant sections in the text as well as any cited reference. Assignments are due the following Friday, or on Dec. 9, whichever comes first. Please write up the starred problems. You are responsible for solutions of the other problems

Your written work reflects your professionalism. Make answers complete and self contained. This means that you should copy or paraphrase each question, provide adequate explanation to help the reader understand the structure of your argument, be thorough in the details, state any theorem that you use and proofread your answer.

Homework from Wednesday to Monday will be due Friday. Late homework that is up to one week late will receive half credit. Homework that is more than one week late will receive no credit at all. Homework that is placed in my mailbox in JWB 228 before about 4:00pm will be considered to be on time.

Please hand in problems A on Friday, August 26.

A. Exercises from Section 1.1 of the text by Strauss:

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4[3, 4*, 5, 6*, 7, 9, 10, 11*]
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1.1.4 Show that the difference of two solutions of an inhomogeneous linear equation $\mathcal{L}u = g$ with the same g is a solution to the homogeneous equation $\mathcal{L}u = 0$.

1.1.6 Are the three vectors $[1, 2, 3]$, $[-2, 0, 1]$ and $[1, 10, 17]$ linearly dependent or independent? Do they span all vectors or not?

1.1.11 Verify that $u(x, y) = f(x)g(y)$ is a solution to the PDE $uu_{xy} = u_x u_y$ for all pairs of (differentiable) functions f and g of one variable.

Please hand in problems B on Friday, September 2.

B. Exercises from Section 1.2,1.3 and 2.1 of the text by Strauss:

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 9[1*, 5*, 7, 9, 11]
18[1*, 10*, 11]
36[1*, 3, 5, 7*]
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1.2.1 Solve the first order equation $2u_t + 3u_x = 0$ with auxiliary condition $u = \sin x$ when $t = 0$.

1.2.5 Solve the equation $\sqrt{1 - x^2}\, u_x + u_y = 0$ with condition $u(0, y) = y$.

1.3.1 Carefully derive the equation of a vibrating string in medium in which the resistance is proportional to the velocity.

1.3.10 If $\mathbf{f} : \mathbf{R}^3 \to \mathbf{R}^3$ is a continuously differentiable and $|\mathbf{f}(\mathbf{x})| \leq \dfrac{1}{1+|\mathbf{x}|^3}$ for all $\mathbf{x} \in \mathbf{R}^3$, show that

$$\iiint_{\mathbf{R}^3} \nabla \bullet \mathbf{f}(\mathbf{x})\, d\mathbf{x} = 0.$$

[Hint: use the meaning of the integral on all of space combined with the divergence theorem on a large ball. Then let its radius go to infinity.]

2.1.1 Solve $u_{tt} - c^2 u_{xx} = 0$, $u(x,0) = e^x$, $u_t(x,0) = \sin x$.

2.1.7 If $\varphi(z)$ and $\psi(z)$ are odd functions of z, show that the solution $u(x,t)$ of the initial value problem on $\mathbf{R} \times [0,\infty)$, $u_{tt} - c^2 u_{xx} = 0$, $u(x,0) = \varphi(x)$, $u_t(x,0) = \psi(x)$ is odd in x for all t.

Please hand in problems C on Friday, September 9.

C. Exercises from Section 1.2, 1.3 and 2.1 of the text by Strauss:

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18[3*, 5]
24[1*, 5*]
40[2, 5*]
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1.3.3 On the sides of a thin rod, heat exchange takes place (obeying Newton's Law of Cooling—flux proportional to temperature difference) with a medium of constant temperature T_0. What is the equation satisfied by temperature $u(x,t)$, neglecting its variation across the rod?

1.4.1 By trial and error, find the solution of the diffusion equation $u_t = u_{xx}$ with the initial condition $u(x,0) = x^2$.

1.4.5 Two homogeneous rods have the same cross section, specific heat c and density ρ but different heat conductivities κ_1 and κ_2 and lengths L_1 and L_2. Let $k_j = \kappa_j/(c\rho)$ be their diffusion constants. They are welded together so that the termperature u and the heat flux κu_x at the weld are continuous. The left-hand rod has its left end maintained at temperature zero. The right-hand rod has its end maintained at temperature T degrees.

(a) Find the equilibrium temperature of the composite rod.

(b) Sketch it as a function of x in casee $k_1 = 2$, $k_2 = 1$, $L_1 = 3$, $L_2 = 2$ and $T = 10$.

2.2.5 For the damped string equation

$$u_{tt} + r u_t - c^2 u_{xx} = 0, \qquad \text{where } r > 0,$$

show that the energy decreases.

Please hand in problems D on Friday, September 16.

D. Exercises from Section 1.5, 1.6 and 2.3 of the text by Strauss:

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27[2, 3*]
31[1, 2*, 5*]
44[2, 3, 4*, 6, 7*]
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1.5.3 Solve the boundary value problem $u'' = 0$ for $0 < x < 1$ with $u'(0) + ku(0) = 0$ and $u'(1) \pm ku(1) = 0$. Do the $+$ and $-$ cases separately. What is special about the case $k = 2$?

1.6.2 Find the regions in the xy-plane where the equation is elliptic, hyperbolic, or parabolic. Sketch them.

$$(1 + x)u_{xx} + 2xyu_{xy} - y^2 u_{yy} = 0$$

1.4.5 Reduce the elliptic equation

$$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u$$

to the form $v_{xx} + v_{yy} + cv = 0$ by a change of dependent variable $u = ve^{\alpha x + \beta y}$ and then a change scale $y' = \gamma y$.

2.3.4 Consider the diffusion equation $u_t = u_{xx}$ in $(0 < x < 1, 0 < t < \infty)$ with $u(0, t) = u(1, t) = 0$ and $u(x, 0) = 4x(1 - x)$.

(a) Show $0 < u(x, t) < 1$ for all $0 < x < 1$ and $0 < t$.

(b) Show that $u(x, t) = u(1 - x, t)$ for all $t \geq 0$ and $0 \leq x \leq 1$.

(c) Use the energy method to show that $\int_0^1 u^2\,dx$ is a strictly decreasing function of t.

2.3.7 (a) If $u_t - ku_{xx} = f$ and $v_t - v_{xx} = g$, $f \leq g$ and $u \leq v$ at $x = 0$, $x = \ell$ and $t = 0$, prove that $u \leq v$ for $0 \leq x \leq \ell$ and $0 \leq t$.

(b) If $v_t - v_{xx} \geq \sin x$ for $0 \leq x \leq \pi$, $0 < t < \infty$, and if $v(0, t) \geq 0$ and $v(\pi, t) \geq 0$ and $v(x, 0) \geq 0$, use part (a) to show that $v(x, t) \geq (1 - e^{-t}) \sin x$.

Please hand in problems E on Friday, September 23.

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A*
50[1*, 5, 6, 7, 9*, 13, 16*, 18*]
54[1*]
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E. Exercises from Sections 1.6, 2.4 and 2.5 of the text by Strauss:

A Classify. Reduce to normal form. Find the general solution.

$$4u_{xx} - 8_{xy} + 4u_{yy} = 1, \qquad 4u_{xx} - 4u_{xy} - 3u_{yy} = 0.$$

2.4.1 Solve the differential equation with initial conditions. Write answer in terms of $\mathcal{E}\mathrm{rf}(\bullet)$.

$$u_t = ku_{xx}, \qquad \text{for } x \in \mathbf{R} \text{ and } t > 0;$$

$$u(x, 0) = \begin{cases} 1, & \text{if } |x| \leq \ell; \\ 0, & \text{if } |x| > \ell. \end{cases}$$

2.4.9 Solve the diffusion equation $u_t = ku_{xx}$ with initial condition $u(x,0) = x^2$ by the following special method. First show that u_{xxx} satisfies the diffusion equation with zero initial condition. Argue that uniqueness implies $u_{xxx} \equiv 0$. Integrating thrice, $u(x,t) = A(t)x^2 + B(t)x + C(t)$. Then solve for A, B and C bu plugging u into the original problem.

2.4.16 Solve the diffusion equation with constant dissipation $b > 0$.

$$\begin{aligned} u_t &= ku_{xx} - bu, && \text{for } x \in \mathbf{R} \text{ and } t > 0; \\ u(x,0) &= \phi(x), && \text{for } x \in \mathbf{R}. \end{aligned}$$

Hint: change variables to $u = e^{-bt}v$.

2.4.18 Solve the diffusion equation with constant convection $V \in \mathbf{R}$.

$$\begin{aligned} u_t &= ku_{xx} - Vu_x, && \text{for } x \in \mathbf{R} \text{ and } t > 0; \\ u(x,0) &= \phi(x), && \text{for } x \in \mathbf{R}. \end{aligned}$$

Hint: go to a moving frame by substituting $y = x - Vt$.

2.5.1 Show that there is no maximum principle for the wave equation.

Please hand in problems F on Friday, September 30.

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58[1*, 2*]
64[5*]
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F. Exercises from Sections 3.1 and 3.2 of the text by Strauss:

3.1.1 Solve on the half-line

$$\begin{aligned} u_t &= ku_{xx}, && \text{for } x > 0 \text{ and } t > 0; \\ u(x,0) &= e^{-x}, && \text{for } x > 0; \\ u(0,t) &= 0, && \text{for } t > 0; \end{aligned}$$

3.1.2 Solve on the half-line

$$\begin{aligned} u_t &= ku_{xx}, && \text{for } x > 0 \text{ and } t > 0; \\ u(x,0) &= 0, && \text{for } x > 0; \\ u(0,t) &= 1, && \text{for } t > 0; \end{aligned}$$

3.1.5 Solve by the reflection method. The solution has a singularity; find its location.

$$\begin{aligned} u_{tt} &= 4u_{xx}, && && \text{for } x > 0 \text{ and } t > 0; \\ u(x,0) &= 1, && u_t(x,0) = 0 && \text{for } x > 0; \\ u(0,t) &= 0, && && \text{for } t > 0; \end{aligned}$$

Please hand in problems G on Friday, October 7.

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58[5, 9]
64[9*]
68[1, 2*]
76[1*, 3, 5*, 11*]
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G. Exercises from Sections 3.1 – 3.4 of the text by Strauss:

3.2.9 Find $u(\frac{2}{3}, 2)$ and $u(\frac{1}{4}, \frac{7}{2})$ if

$$\begin{aligned} &u_{tt} = u_{xx}, && \text{for } 0 < x < 1 \text{ and } 0 < t;\\ &u(x,0) = x^2(1-x), \qquad u_t(x,0) = (1-x)^2 && \text{for } 0 < x < 1;\\ &u(0,t) = 0, \qquad u(1,t) = 0, && \text{for } t > 0; \end{aligned}$$

3.3.2 Solve completely the inhomogeneous diffusion problem on the half-line by carrying out the subtraction method begun in the text.

$$\begin{aligned} v_t - kv_{xx} &= f(x,t), && \text{for } x > 0 \text{ and } t > 0;\\ v(x,0) &= \varphi(x), && \text{for } x > 0;\\ v(0,t) &= h(t), && \text{for } t > 0; \end{aligned}$$

3.4.1 Solve

$$\begin{aligned} &u_{tt} = c^2 u_{xx} + xt, && \text{for } -\infty < x < \infty \text{ and } 0 < t;\\ &u(x,0) = 0, \qquad u_t(x,0) = 0 && \text{for } -\infty < x < \infty; \end{aligned}$$

3.4.5 Let $f(x,t)$ be any function and

$$u(x,t) = \frac{1}{2c} \iint_{\Delta} f(y,s)\, dy\, ds$$

where Δ is the triangle of dependence. Verify directly by differentiation that

$$\begin{aligned} &u_{tt} = c^2 u_{xx} + f(x,t), && \text{for } -\infty < x < \infty \text{ and } 0 < t;\\ \text{and} \qquad &u(x,0) = 0, \qquad u_t(x,0) = 0 && \text{for } -\infty < x < \infty. \end{aligned}$$

Hint: write the formula as iterated integral and use the theorem in Appendix 3:

$$u(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\, dy\, ds.$$

3.4.11 Show by direct substitution that

$$u(x,t) = \begin{cases} h\left(t - \dfrac{x}{c}\right), & \text{for } x < ct;\\ 0, & \text{for } x \geq ct; \end{cases}$$

solves IBVP for the wave equation on the half line

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, && \text{for } x > 0 \text{ and } t > 0;\\ &u(x,0) = 0, \qquad u_t(x,0) = 0 && \text{for } x > 0;\\ &u(0,t) = h(t), && \text{for } t > 0. \end{aligned}$$

Please hand in problems H on Friday, October 21.

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87[1, 2*, 3*, 4*, 5]
90[1, 2*, 3*]
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H. Exercises from Sections 4.1 – 4.2 of the text by Strauss:

4.1.2 Consider a metal rod ($0 < x < \ell$), insulated along the sides but not at its ends, which is initially at temperature $= 1$. Suddenly both ends are plunged into a bath of temperature $= 0$. Write the differential equation, boundary conditions and initial condition. Write the formula for the temperature $u(x,t)$ at later times. You may assume the infinite series expansion

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{\ell} + \frac{1}{3}\sin\frac{3\pi x}{\ell} + \frac{1}{5}\sin\frac{5\pi x}{\ell} + \cdots\right)$$

4.1.3 A quantum-mechanical particle on the line with infinite potential outside the interval $(0,\ell)$ "particle in a box" solves Schrödinger's Equation $u_t = iu_{xx}$ on $(0,\ell)$ with Dirichlet boundary conditions at the ends. Separate variables and represent the solution as a series using

$$\lambda_n = \frac{\pi^2 n^2}{\ell^2}, \qquad X_n(x) = \sin\frac{\pi n x}{\ell}, \qquad n = 1,2,3,\ldots$$

4.1.4 Consider waves in a resistant medium which satisfy the equation with constant r such that $0 < \ell r < 2\pi c$. Write down the series expansion of the solution.

$$\begin{aligned} &u_{tt} = c^2 u_{xx} - r u_t, && \text{for } 0 < x < \ell \text{ and } 0 < t;\\ &u(0,t) = 0, \quad u(\ell,t) = 0, && \text{for } 0 < t;\\ &u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) && \text{for } 0 < x < \ell. \end{aligned}$$

4.2.2 Separate variables in the wave equation with Neumann boundary condition on the left and Dirichlet boundary condition on the right.

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, && \text{for } 0 < x < \ell \text{ and } 0 < t;\\ &u_x(0,t) = 0, \quad u(\ell,t) = 0, && \text{for } 0 < t;\\ &u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) && \text{for } 0 < x < \ell. \end{aligned}$$

(a) Show that the eigenfunctions are $\cos\dfrac{\pi(n+\frac{1}{2})x}{\ell}$ for $n = 0,1,2,3,\ldots$

(b) Write the series expansion for the solution $u(x,t)$.

4.2.3 Consider diffusion inside a closed circular tube whose length (circumference) is 2ℓ. Let x denote the arclength parameter, where $-\ell \le x \le \ell$. Then the concentration of the diffusing substance satisfies diffusion equation with periodic boundary conditions

$$\begin{aligned} &u_t = k u_{xx}, && \text{for } -\ell \le x \le \ell \text{ and } 0 < t;\\ &u(-\ell,t) = u(\ell,t), \quad u_x(-\ell,t) = u_x(\ell,t), && \text{for } 0 < t;\\ &u(x,0) = \varphi(x) && \text{for } -\ell \le x \le \ell. \end{aligned}$$

(a) Show that the eigenvalues are $\lambda_n = \dfrac{\pi^2 n^2}{\ell^2}$ for $n = 0,1,2,3,\ldots$.

(b) Define a_n and b_n and show that the concentration has the expansion

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{\pi n x}{\ell} + b_n \sin \frac{\pi n x}{\ell} \right\} \exp\left(-\frac{k\pi^2 n^2 t}{\ell^2} \right).$$

Please hand in problems I on Friday, October 28.

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 97[1*, 2, 3, 4*, 5, 6*, 11*, 16]
107[2, 3, 4*, 5, 9*]
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I. Exercises from Sections 4.3 and 5.1 of the text by Strauss:

4.3.1 Assuming $a \neq 0$, find the eigenvalues graphically for the eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = 0, \qquad X'(\ell) + aX(\ell) = 0.$$

4.3.4 Consider the Robin eigenvalue problem

$$X'' + \lambda X = 0, \qquad X'(0) - a_0 X(0) = 0, \qquad X'(\ell) + a_1 X(\ell) = 0$$

for

$$a_0 < 0, \qquad a_1 < 0, \qquad -a_0 - a_1 < a_0 a_1 \ell.$$

Show that there are two negative eigenvalues. This case maybe called "substantial absorption at both ends." [Hint: Show that the graph of $g(\gamma) = (a_0 + a_1)\gamma/(\gamma^2 + a_0 a_1)$ has a single maximum and crosses the line $y = 1$ in two places. Deduce that it crosses the hyperbolic tangent curve in two places.]

4.3.6 Let $a_0 = a_1 = a$ in the Robin eigenvalue problem

$$X'' + \lambda X = 0, \qquad X'(0) - a_0 X(0) = 0, \qquad X'(\ell) + a_1 X(\ell) = 0.$$

(a) Show that there are no negative eigenvalues if $a > 0$, there is one if $-2/\ell < a < 0$ and two if $a < -2/\ell$.

(b) Show that zero is an eigenvalue if and only if $a = 0$ or $a = -2/\ell$.

4.3.11 Consider the total energy for the wave equation

$$E = \frac{1}{2} \int_0^\ell \frac{u_t^2}{c^2} + u_x^2 \, dx.$$

(a) Prove that the energy is conserved assuming the Dirichlet boundary conditions.

(b) Do the same for Neumann boundary conditions.

(c) For the Robin boundary conditions show that

$$E_R = E + \tfrac{1}{2} a_1 u^2(\ell, t) + \tfrac{1}{2} a_0 u^2(0, t)$$

is conserved. Thus, while the total energy E_R is constant, show that some of the internal energy E is "lost" to the boundary if a_0 and a_1 are positive and "gained" if a_0 and a_1 are negative.

5.1.4 Find the Fourier cosine series of the function $|\sin x|$ on the interval $(-\pi, \pi)$. Use it to find the sums

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}.$$

5.1.9 Solve

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, && \text{for } 0 < x < \pi \text{ and } 0 < t; \\ u(x,0) &= 0, && \\ u_t(x,0) &= \cos^2 x, && \text{for } 0 < x < \pi; \\ u_x(0,t) &= 0, && \\ u_x(\pi,t) &= 0, && \text{for } 0 < t. \end{aligned}$$

Please hand in problems J on Friday, November 4.

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113[1, 3, 4, 6*, 7, 11*, 17]
118[2*, 3*, 5, 6, 8, 9*, 15*]
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J. Exercises from Sections 5.2 and 5.3 of the text by Strauss:

5.2.2 Show that the cosine series on $(0,\ell)$ can be derived from the full Fourier Series on $(-\ell,\ell)$ by using the even extension of a function.

5.2.11 Find the full Fourier series of e^x on $(-\ell,\ell)$ in its real and complex forms. [Hint: find the complex form first.]

5.3.2 (a) On the interval $(-\ell,\ell)$, show that the function x is orthogonal to the constant function.

(b) Find a quadratic poynomial that is orthogonal to both 1 and x.

(c) Find a cubic polynomial that is orthogonal to all quadratics.

5.3.3 Find the solution explicitly in series form.

$$\begin{aligned} &\text{P. D. E.} & u_{tt} &= c^2 u_{xx}, && \text{for } 0 < x < \ell \text{ and } 0 < t; \\ &\text{I. C.} & u(x,0) &= x, && \\ & & u_t(x,0) &= 0, && \text{for } 0 < x < \ell; \\ &\text{B. C.} & u(0,t) &= 0, && \\ & & u_x(\ell,t) &= 0, && \text{for } 0 < t. \end{aligned}$$

5.3.9 For solutions f and g, the boundary conditions on the interval (a,b) are *symmetric* if

$$\Big[f'(x)g(x) - f(x)g'(x)\Big]_{x=a}^{b} = 0.$$

Show that

$$X(b) = \alpha X(a) + \beta X'(a) \qquad \text{and} \qquad X'(b) = \gamma X(a) + \delta X'(a)$$

are symmetric if and only if $\alpha\delta - \beta\gamma = 1$.

5.3.15 Show that all eigenvalues are real and nonnegative

$$X'''' = \lambda X, \qquad X(0) = X(\ell) = X''(0) = X''(\ell) = 0.$$

Please hand in problems K on Friday, November 4.

`129[1, 2, 3*, 4, 5, 6*, 9, 12*, 15*]`

K. Exercises from Sections 5.4 of the text by Strauss:

5.4.3 Let γ_n be a sequence of numbers tending to infinity. Let $\{f_n\}$ be the sequence of functions defined on $[-1, 2]$ as follows

$$f_n(x) = \begin{cases} \gamma_n, & \text{if } \frac{1}{2} - \frac{1}{n} \leq x < \frac{1}{2}; \\ -\gamma_n, & \text{if } \frac{1}{2} < x \leq \frac{1}{2} + \frac{1}{n}; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that $f_n(x) \to 0$ pointwise.

(b) Show that the convergence is not uniform.

(c) Show that $f_n(x) \to 0$ in the $\mathcal{L}^2$-sense if $\gamma_n = n^{\frac{1}{3}}$.

(d) Show that $f_n(x)$ does not converge in the $\mathcal{L}^2$-sense if $\gamma_n = n$.

5.4.6 Find the sine series of the function $\cos x$ on the interval $(0, \pi)$. For each x satisfying $-\pi \leq x \leq \pi$, what is the sum of the series?

5.4.12 Start with the Fourier sine sreies for $f(x) = x$ on the interval $(0, \ell)$. Apply Parseval's identity. Find the sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

5.4.15 Let $\phi(x) = 1$ for $0 < x < \pi$. Expand

$$1 \sim \sum_{n=0}^{\infty} B_n \sin\left(\left(n + \tfrac{1}{2}\right) x\right).$$

(a) Find B_n.

(b) Let $-2\pi < x < 2\pi$. For which such x does the series converge? For each such x, what is the sum of the series? [Hint: Think of extending $\phi(x)$ beyond the interval $(0, \pi)$.]

(c) Apply Parseval's equality to this series. Use it to calculate the sum

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$$

Please hand in problems L on Friday, November 11.

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139[1, 2*, 8*]
144[1*, 2*, 9*]
154[4*]
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L. Exercises from Sections 5.5, 5.6 and 6.1 of the text by Strauss:

5.5.2 Prove the Schwarz Inequality for any pair of functions f and g in a inner product space.

$$|\,(f,g)\,|^2 \le \|f\|^2\,\|g\|^2.$$

(Hint: Consider the expression $\|f+tg\|^2$ where t is a scalar. Use the value t where the expression is minimum.)

5.5.8 Prove that both integrals in the equation tend to zero. Assume that f is a 2π-periodic function such that f and f' are piecewise continuous. Then the integrals are

$$S_N(x) - \frac{f(x+)+f(x-)}{2} = \int_0^\pi g_+(s)\sin[(N+\tfrac{1}{2})s]\,ds + \int_{-\pi}^0 g_-(s)\sin[(N+\tfrac{1}{2})s]\,ds$$

where

$$g_\pm(s) = \frac{f(x+s)-f(x\pm)}{\sin\frac{s}{2}}.$$

5.6.1 Consider the initial-boundary value problem for the heat equation.

P. D. E.	$u_t = u_{xx},$	for $0<x<1$ and $0<t$;
I. C.	$u(x,0) = x^2,$	for $0<x<1$;
B. C.	$u_x(0,t) = 0,$	
	$u(1,t) = 1,$	for $0<t$.

(a) Solve as a series. Compute the first two coefficients explicitly.

(b) What is the equilibrium state (the term that does not go to zero)?

5.6.2 For the problem, complete the calculation for the series in case $j(t)=0$ and $h(t)=e^t$.

P. D. E.	$u_t = ku_{xx},$	for $0<x<\ell$ and $0<t$;
I. C.	$u(x,0) = 0,$	for $0<x<\ell$;
B. C.	$u(0,t) = h(t),$	
	$u(\ell,t) = j(t),$	for $0<t$.

5.6.9 Use the method of subtraction to solve

P. D. E	$u_{tt} = 9u_{xx},$	for $0<x<1$ and $0<t$;
I. C.	$u(x,0) = 0,$	
	$u_t(x,0) = 0,$	for $0<x<1$;
B. C.	$u(0,t) = h,$	
	$u(1,t) = k,$	for $0<t$.

where h and k are given constants.

6.1.4 For constants a, b, A, B, solve the BVP for Laplace's Equation in the spherical shell

$$\begin{array}{lrl} \text{P. D. E.} & u_{xx} + u_{yy} + u_{zz} = 0, & \text{for } 0 < a < r < b; \\ \text{B. C.} & u = A, & \text{for } r = a, \\ & u = B, & \text{for } r = b. \end{array}$$

(Hint: Look for solutions depending only on the radial variable r.)

Please hand in problems M on Wednesday, November 23.

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154[1, 2, 9, 10*, 11]
158[1*, 3, 4*, 6, 7]
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M. Exercises from Sections 6.1 and 6.2 of the text by Strauss:

6.1.10 Prove the uniqueness of the Dirichlet problem for the Poisson equation on a nice domain $\mathcal{D} \in \mathbf{R}^d$ by the energy method. (Hint: Multiply the difference of two solutions $w = u - v$ by Δw and use the divergence theorem.)

$$\begin{array}{lrl} \text{P. D. E} & \Delta u = f, & \text{in } \mathcal{D}; \\ \text{B. C.} & u = g, & \text{in } \partial\mathcal{D}. \end{array}$$

6.2.1 Solve the BVP for Laplace's equation on the rectangle. (Hint: note that the necessary condition for Neumann boundary values is satisfied. A shortcut is to guess that the solution is given by a quadratic polynomial.)

$$\begin{array}{lrl} \text{P. D. E} & u_{xx} + u_{yy} = 0, & \text{for } 0 < x < a \text{ and } 0 < y < b; \\ \text{B. C.} & u_x(0, y) = -a, & \\ & u_x(a, y) = 0, & \text{for } 0 < y < b, \\ & u_y(x, 0) = b, & \\ & u_y(x, b) = 0, & \text{for } 0 < x < a. \end{array}$$

6.2.4 Solve the BVP for Laplace's equation on the rectangle.

$$\begin{array}{lrl} \text{P. D. E} & u_{xx} + u_{yy} = 0, & \text{for } 0 < x < 1 \text{ and } 0 < y < 1; \\ \text{B. C.} & u_x(0, y) = 0, & \\ & u_x(1, y) = y^2, & \text{for } 0 < y < 1, \\ & u(x, 0) = x, & \\ & u(x, 1) = 0, & \text{for } 0 < x < 1. \end{array}$$

Please hand in problems N on Friday, December 2.

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163[1*, 2*]
167[1*, 2, 4, 5a*, 7, 9, 10*, 11, 12]
147[3, 5*]
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N. Exercises from Sections 6.3, 6.4 and 7.1 of the text by Strauss:

6.3.1 Suppose that u is harmonic in the disk $\mathcal{D}$ $x^2 + y^2 < 4$, continuous on $\overline{\mathcal{D}}$ and equals $3\sin 2\theta + 1$ rot $r = 2$. Eithout finding the solution, answer the following questions.

(a) Find the maximum value of u on $\overline{\mathcal{D}}$.

(b) Calculate the value of u at the origin.

6.3.2 Solve Laplace's Equation on the disk

$$\begin{array}{lll} \text{P. D. E} & u_{xx} + u_{yy} = 0, & \text{for } r < a; \\ \text{B. C.} & u = 1 + 3\sin\theta, & \text{for } r = a. \end{array}$$

6.4.1 Solve Laplace's Equation on the exterior of the disk

$$\begin{array}{lll} \text{P. D. E} & u_{xx} + u_{yy} = 0, & \text{for } r > a; \\ \text{B. C.} & u = 1 + 3\sin\theta, & \text{for } r = a. \\ & u \text{ is bounded}, & \text{as } r \to \infty. \end{array}$$

6.4.5a Find the steady state temperature distribution inside an annular plate $(1 < r < 2)$ whose outer edge $(r = 2)$ is insulated and on whose inner edge $(r = 1)$ the temperature is maintained as $\sin^2\theta$. (Find explicitly all coefficients, *etc.*

6.4.10 Solve Laplace's Equation on the quarter disk

$$\begin{array}{lll} \text{P. D. E} & u_{xx} + u_{yy} = 0, & \text{for } r < a,\ x > 0,\ y > 0; \\ \text{B. C.} & u = 0, & \text{on } x = 0 \text{ and on } y = 0, \\ & \dfrac{\partial u}{\partial n} = 1, & \text{on } r = a. \end{array}$$

7.1.5 Prove Dirichlet's principle for Neumann boundary conditions. It asserts that among all real valued functions $w(x)$ on $\mathcal{D}$, the quantity

$$E[w] = \frac{1}{2}\int_{\mathcal{D}} |\nabla w|^2\, dV - \int_{\partial\mathcal{D}} hw\, dA$$

is smallest for $w = u$, where u is the solution of the Neumann Problem

$$\begin{array}{lll} \text{P. D. E} & \Delta\, u = 0, & \text{in } \mathcal{D}; \\ \text{B. C.} & \dfrac{\partial u}{\partial n} = h, & \text{on } \partial\mathcal{D}. \end{array}$$

It is required that the average on the boundary of the given function h is zero.