Math 5470 § 1. **Final Exam** Name: Practice Problems
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1. *Give an example of a system which simultaneously posseses the following two properties.*

   (a) it is dissipative, it i.e., any volume in phase space contracts under the flow;

   (b) "almost all" trajectories go to infinity as time $t \to \infty$.

   Consider the linear system

$$\dot{x} = ax$$
$$\dot{y} = by$$

   the system is dissipative if the divergence of the vector field is everywhere negative. Computing

$$\operatorname{div}(ax, by) = a + b$$

   The solution of the system with $(x(0), y(0)) = (x_0, y_0)$ is

$$x(t) = x_0 e^{at}, \qquad y(t) = y_0 e^{bt}$$

   For almost all solutions to tend to infinity, we need for almost all (x_0, y_0) that

$$|(x(t), y(t))|^2 = x_0^2 e^{2at} + y_0^2 e^{2bt} \to \infty \qquad \text{as } t \to \infty$$

   Thus we need $a + b < 0$ and one or the other $a > 0$ and $x_0 \neq 0$ or $b > 0$ and $y_0 \neq 0$. For example, taking $a = 1$, $b = -2$ we have $a + b = -1$ so dissipative and

$$|(x(t), y(t))|^2 = x_0^2 e^{2t} + y_0^2 e^{-4t} \to \infty \qquad \text{as } t \to \infty$$

   whenever $x_0 \neq 0$, which is almost everywhere.

2. *What does it mean that the Lorenz System exhibits* sensitive dependence on initial conditions? *What is the Lorenz map? How did Lorenz argue that the attractor in his system is not just one long stable periodic orbit?*

   "Sensitive dependence on initial conditions" means that if a trajectory starts on the attractor there is another trajectory that starts very close to it such that the two will rapidly diverge from each other, and thereafter have totally different futures. More precisely, after the transients have decayed so that $x(t)$ is close to the attractor at time t, consider the nearby point $x(t) + \delta(t)$ where $|\delta(t)| = \delta_0$ is very small. Sensitive dependence means that it is true that there is $\delta(0)$ and $c, \lambda > 0$ such that

$$|\delta(t + s)| \geq c\delta_0 e^{\lambda s}$$

   over a range $0 \leq s \leq S$. Numerical studies show that for the Lorenz System, λ, the Liapunov Exponent is about 0.9. The $\ln |\delta(t + s)|$ actually oscillates about a line of slope λ. This growth rate stops when the oscillations reach the scale of the trapping ball.

   Let z_n denote consecutive relative maxima of the z component of a solution to the Lorenz Equations. Lorenz observed that when many (z_n, z_{n+1})'s are plotted on the plane, then the picture is almost that of a curve $z_{n+1} = f(z_n)$. The Lorenz map is the function $f : \mathbf{R} \to \mathbf{R}$. The map is tent-shaped with slope $|f'(z)| > 1$ at all z (except at the kink where f' does not exist.)

Lorenz' argument that there are no periodic stable orbits goes as follows. Suppose that there is a periodic orbit $(x(t), y(t), z(t))$ that has k maxima of $z(t)$ in a period. He argues that it cannot be stable. Thus, assuming the Lorenz map tells the story, we have

$$z_1 = f(z_0), \qquad z_2 = f(z_1), \qquad \cdots \qquad z_0 = z_k = f(z_{k-1}).$$

Lets check the stability of this orbit. The maximum at a perturbation $z_1+\eta_1$ is approximated by the linearization

$$\begin{aligned} z_1 + \eta_1 &= f(z_0 + \eta_0) \approx f(z_0) + f'(z_0)\eta_0 \\ z_{j+1} + \eta_{j+1} &= f(z_j + \eta_j) \approx f(z_j) + f'(z_j)\eta_j \end{aligned}$$

so that $\eta_{j+1} \approx f'(z_j)\eta_j$ for all j. It follows that

$$\eta_k \approx \left(\prod_{i=0}^{k-1} f'(z_i) \right) \eta_0$$

so in norm, if $b = \min\{|f'(z_0)|, |f'(z_i)|, \ldots, |f'(z_k)|\} > 1$ then for all ℓ,

$$|\eta_\ell| \approx \left(\prod_{i=0}^{\ell-1} |f'(z_i)| \right) |\eta_0| \geq b^\ell |\eta_0|$$

which proves that the periodic orbit is unstable.

3. *Consider the iterated map with parameter* $-\infty < \mu < \infty$

$$x_{n+1} = \mu x_n + 2$$

Does it exhibit sensitive dependence on initial conditions? Is it chaotic?

Suppose that y_n is another solution with y_0 close to x_0. Consider the equation for the difference

$$w_{n+1} = x_{n+1} - y_{n+1} = (\mu x_n + 2) - (\mu y_n + 2) = \mu(x_n - y_n) = \mu w_n$$

whose solution is

$$w_n = (x_0 - y_0)\mu^n$$

It follows that the difference grows exponentially or has sensitive dependence to initial conditions if $|\mu| > 1$. In fact we can write an explicit solution of the difference equation. If $\mu = 1$ we have

$$x_1 = x_0 + 2, \quad x_2 = x_1 + 2 = x_0 + 4, \quad \ldots \quad x_n = x_0 + 2n$$

Otherwise $\mu \neq 1$ and

$$x_1 = \mu x_0 + 2, \qquad x_2 = \mu x_1 + 2 = \mu^2 x_0 + 2\mu + 2, \qquad x_3 = \mu^3 x_0 + 2\mu^2 + 2\mu + 2, \ldots$$

Hence we have a closed formula

$$x_n = \mu^n x_0 + \frac{2(1-\mu^n)}{1-\mu} = f^n(x_0)$$

To see that it is correct, we check

$$
\begin{aligned}
x_0 &= \mu^0 x_0 + \frac{2(1-\mu^0)}{1-\mu} = x_0 + 0 \\
\mu x_n + 2 &= \mu\left(\mu^n x_0 + \frac{2(1-\mu^n)}{1-\mu}\right) + 2 = \mu^{n+1} x_0 + \frac{2\mu(1-\mu^n) + 2(1-\mu)}{1-\mu} \\
&= \mu^{n+1} x_0 + \frac{2(1-\mu^{n+1})}{1-\mu} = x_{n+1}
\end{aligned}
$$

Thus we see that if $|\mu| < 1$ then $x_n \to \mu/(1-\mu)$ converges as $n \to \infty$. If $\mu = -1$ then x_n is periodic of period 2. Otherwise, if $\mu < -1$ or $\mu \ge 1$, the solution tends to infinity, which is regarded as converging to a point. Thus the motion is not chaotic: it converges to a point or is periodic.

4. *Suppose that $y(x)$ satisfies the differential equation*

$$\dot{y} = f(x, y)$$

where $f(x, y)$ is a smooth function on the plane that satisfies $f(x, a) \ge 0$ and $f(x, b) \le 0$ for all x. Suppose that $a \le y(0) \le b$. Show that the solution stays bounded $a \le y(x) \le b$ for all x, and thus exists for all x

If we can show the solution remains bounded, then the solution exists for all time because the the only way it could have stopped at a finite time is if it had blown up. If a solution remains bounded, then the local existence theorem tells us that a solution could be continued beyond any x.

It remains to argue that if the solution exists on the interval $0 \le x \le T$, then $a \le y(x) \le b$ for all $0 \le x \le T$. Intuitively, $y = a$ and $y = b$ are lower and upper barriers. To show $y(x) \le b$, if a solution crosses $y = b$ from below then y is non-decreasing. However the assumption is only that $\dot{y} \le 0$ which says only that it is non-increasing when $y = b$.

But if we knew the sharper inequality that $f(x, b) < 0$ for all x then the argument would be complete. If $y(x_1) > b$ at some $0 < x_1 \le T$ then there is a largest number $x_2 \in [0, x_1)$ such $y(x) < b$ for all $x < x_2$ and $y(x_2) = b$. At this point the function is non-decreasing so $\dot{y}(x_2) \ge 0$, which contradicts $\dot{y}(x_2) = f(x_2, b) < 0$.

Here is where analysis comes in. We perturb the equation so that it has stronger inequalities at the barriers and then approximate the desired solutions by solutions of the perturbed equations. This argument uses continuous dependence on a parameter. The modified differential equation for parameter $\epsilon > 0$ is

$$\dot{y}_\epsilon = f(x, y_\epsilon) + \epsilon(a + b - 2y_\epsilon) = f(x, y_\epsilon; \epsilon)$$

Note that $f(x, a; \epsilon) > 0$ and $f(x, b; \epsilon) < 0$. Thus by the strict inequality, we have $a \le y_\epsilon(x) \le b$ for all $0 \le x \le T$. Now by continuous dependence, since $f(x, y; \epsilon) \to f(x, y)$ uniformly on $[0, T] \times [a, b]$ as $\epsilon \to 0+$ we have uniform convergence on $0 \le x \le T$

$$y(x) = \lim_{\epsilon \to 0+} y_\epsilon(x)$$

which is in the interval $[a, b]$ because all $y_\epsilon(x)$ are.

5. *When the second order equation for small parameter $0 \le \epsilon$*

$$\ddot{x} + \epsilon \dot{x} + x = 0 \tag{1}$$

is written as a system, we get

$$\dot{x} = y$$
$$\dot{y} = -x - \epsilon y$$

Take the positive x-axis as the Poincaré section. Find the corresponding Poincaré map. [Prof. Balk's final.]

The roots of the characteristic poynomial $\lambda^2 + \epsilon\lambda + 1 = 0$ are are

$$\frac{-\epsilon \pm i\sqrt{4-\epsilon^2}}{2}$$

The general solution of (1) is

$$x(t) = \left(A\cos\left(\frac{\sqrt{4-\epsilon^2}}{2}t\right) + B\sin\left(\frac{\sqrt{4-\epsilon^2}}{2}t\right)\right)e^{-\frac{\epsilon}{2}t}$$

Hence

$$x(0) = A; \qquad \dot{x}(0) = -\frac{\epsilon}{2}A + \frac{\sqrt{4-\epsilon^2}}{2}B$$

Thus solution of the initial value problem $x(0) = x_0$, $\dot{x}(0) = y_0 = 0$ is

$$x(t) = \left(x_0\cos\left(\frac{\sqrt{4-\epsilon^2}}{2}t\right) + \frac{\epsilon x_0}{\sqrt{4-\epsilon^2}}\sin\left(\frac{\sqrt{4-\epsilon^2}}{2}t\right)\right)e^{-\frac{\epsilon}{2}t}$$

The solution returns to the section when

$$\frac{\sqrt{4-\epsilon^2}}{2}T = 2\pi \qquad \text{or} \qquad T = \frac{4\pi}{\sqrt{4-\epsilon^2}}$$

Thus the Poincaré first return map for $x_0 \geq 0$ is

$$P(x_0;\epsilon) = x(T) = x_0\exp\left(-\frac{2\pi\epsilon}{\sqrt{4-\epsilon^2}}\right)$$

Note that $\frac{\partial}{\partial\epsilon}P(x_0;0) = -x_0\pi$

Perhaps Prof. Balk wanted to solve the Poincaré map perturbatively. In other words, the solution would be a perturbation of $\epsilon = 0$ equation. In this case, the $\epsilon = 0$ solution for the initial data $(r, 0)$ and Poincaré map is

$$x(t) = r\cos t; \qquad y(t) = -r\sin t; \qquad P(r;0) = x(2\pi) = r$$

Now consider an infinitesimal perturbation $x_\epsilon(t) = r\cos t + \xi(t)$, $y_\epsilon(t) = -r\sin t + \eta(t)$. Plugging into the equations yields

$$\begin{aligned} \dot{x}_\epsilon &= -r\sin t + \dot{\xi} &&= -r\sin t + \eta \\ \dot{y}_\epsilon &= -r\cos t + \dot{\eta} &&= -r\cos t - \xi - \epsilon(-r\sin t + \eta(t)) \end{aligned}$$

Dropping quadratic terms in the perturbation results in the system

$$\dot{\xi} = \eta$$
$$\dot{\eta} = -\xi + \epsilon r\sin t$$

which is the resonantly forced harmonic oscillator

$$\ddot{\xi} + \xi = \epsilon r \sin t$$

Its solution for $\xi(0) = \eta(0) = 0$ is

$$\xi(t) = \frac{\epsilon r}{2}(\sin t - t \cos t)$$

Thus the Poincaré map up to first order is

$$P(r) = x_\epsilon(2\pi) = r \cos 2\pi + \xi(2\pi) = r - \pi r \epsilon$$

which is the same up to the first order in ϵ as the nonlinear solution.

6. *Find and classify the fixed points of the map. For what value of the parameter r is there a superstable fixed point?*

$$x_{n+1} = x_n^2 - r x_n$$

The fixed points satisfy $x = x^2 - rx$ or

$$x = 0 \qquad \text{or} \qquad x = r + 1$$

$f' = 2x - r$. At zero, $f'(0) = -r$ which is stable if $|r| < 1$ and unstable if $|r| > 1$. At $x = r + 1$, $f'(r + 1) = 2(r + 1) - r = r + 2$. This fixed point is stable if $|f'(r + 1)| < 1$ or $|r + 2| < 1$ or $-3 < r < -1$. It is unstable if $|r + 2| > 1$. the remaining cases are marginal and require further analysis.

The fixed points are superstable if $f' = 0$. At the origin, this occurs if $r = 0$ and at $r + 1$ this occurs if $r = -2$.

7. *Consider the nonlinear system. Observe that $\gamma(t) = (\cos t, \sin t, 0)$ is a periodic solution to the system. Use a linearized analysis to determine whether this solution is stable.*

$$\begin{aligned}
\dot{x} &= -y + xz^2 \\
\dot{y} &= x + yz^2 \\
\dot{z} &= -z(x^2 + y^2)
\end{aligned}$$

We need to determine the eigenvalues of the linearized Poincaré map about γ. Take the section $\Sigma = \{(x, y, z) : y = 0, x > 0\}$. Let us compute the variation equation as in Examples 8.7.3 and 8.7.4. Consider an infinitesimal perturbation of of the solution

$$(x, y, z) = (\cos t + \xi(t), \sin t + \eta(t), \zeta(t))$$

where (ξ, η, ζ) are infinitesimal. Substituting

$$\begin{aligned}
-\sin t + \dot{\xi} &= -\sin t - \eta + (\cos t + \xi)\zeta^2 \\
\cos t + \dot{\eta} &= \cos t + \xi + (\sin t + \eta)\zeta^2 \\
\dot{\zeta} &= -\zeta[(\cos t + \xi)^2 + (\sin t + \eta)^2]
\end{aligned}$$

Neglecting quadratic terms in the infinitesimals,

$$\begin{aligned}
\dot{\xi} &= -\eta \\
\dot{\eta} &= \xi \\
\dot{\zeta} &= -\zeta
\end{aligned}$$

whose solution is

$$(\xi(t), \eta(t), \zeta(t)) = (\xi_0 \cos t - \eta_0 \sin t, \eta_0 \cos t + \xi_0 \sin t, \zeta_0 e^{-t})$$

Thus the perturbation of γ, $(\xi_0, 0, \zeta_0) \in \Sigma$ first returns to Σ at the perturbation of γ given by the differential of the Poincaré map

$$DP[(1,0)](\xi_0, \zeta_0) = (\xi(2\pi), \zeta(2\pi)) = (\xi_0, e^{-2\pi}\zeta_0)$$

The eigenvalues are $\lambda = 1, e^{-2\pi}$, thus γ has inconclusive linearized stability: the norms of both eigenvalues are not less than one nor both greater than one.

8. *Using a Poincaré map, show that the system has at least two periodic solutions. Can you determine their stability? Regard the equation as a vector field on the cylinder. Sketch the nullclines and thereby infer the shape of certain key trajectories that can be used to bound the periodic solutions. For instance, sketch the trajectory throught the point $(t, \theta) = (\frac{\pi}{2}, \frac{\pi}{2})$. [Strogatz 8.7.5]*

$$\dot{\theta} + \sin\theta = \sin t$$

Write the forced equation as a system using $x = t$ and $y = \theta$

$$\begin{aligned} \dot{x} &= 1 && = f(x,y) \\ \dot{y} &= \sin x - \sin y && = g(x,y) \end{aligned}$$

Now consider a shooting approach from the lines $x = 0$ to $x = 2\pi$. Denote the solution starting at $(0, y_0)$ by $(x(t), y(t))$ so $x(t) = t$. The Poincaré map is $P(y_0) = y(2\pi)$. A periodic solution satisfies $P(y_0) = y_0$. On the $g = 0$ nullcline,

$$0 = g(x,y) = \sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

whose solutions are the lines $x - y = 2\pi k$ and $x + y = \pi + 2\pi j$ where k and j are integers that divide the plane into a diagonal checkerboard in which g alternates signs.

The trajectory y_1 with $y_1(\frac{\pi}{2}) = \frac{\pi}{2}$ has $\dot{y} = 0$ there and has $\dot{y} < 0$ in the neighboring squares bounded by $x - y = 0$ and $x + y = \pi$. Thus for $0 \le x < \frac{\pi}{2}$, $y_1 \ge \frac{\pi}{2}$ because $\dot{y}_1 \le 0$ and it cannot cross $y = \pi - x$ because it would require $\dot{y}_1 \le -1$ but $g = 0$ on this curve. It follows that $\frac{\pi}{2} \le y_1(0) \le \pi$. By the same token, for $\frac{\pi}{2} \le x \le 2\pi$ we have $y_1(x) \le \frac{\pi}{2}$ because $\dot{y}_1 < 0$ and it cannot dip below $y = \pi - x$ because to cross it would need $\dot{y}_1 \le -1$ but $g = 0$ on this curve. If y_1 crosses $y = x - 2\pi$ at some $\frac{3\pi}{2} \le x_2 \le 2\pi$, then $-\frac{\pi}{2} \le y_1 \le x - 2\pi$ because $\dot{y}_2 \ge 0$ for $x_2 \le x \le 2\pi$ and it cannot cross $y = x - 2\pi$ from below because it would require $\dot{y}_1 \ge 1$ but $g = 0$ there. The upshot is that $-\frac{\pi}{2} \le y_1(2\pi) \le \frac{\pi}{2}$.

Observe that the system is invariant under the transformation

$$x \to x + \pi, \qquad y \to -y$$

Thus there is a solution y_2 with $y_2(\frac{3\pi}{2}) = -\frac{\pi}{2}$ which satisfies $-\pi \le y_2(\pi) \le -\frac{\pi}{2}$ by translation. In particular $y_2(\pi) < y_1(\pi)$ so $y_2(x) < y_1(x)$ for all $0 \le x \le 2\pi$. Arguing as before, y_2 cannot cross either line $y = x - \pi$ nor $y = \pi - x$ and is increasing in the squares to the left and right of $(\frac{3\pi}{2}, \frac{\pi}{2})$. Thus it satisfies $-\frac{3\pi}{2} \le y_2 \le -\frac{\pi}{2}$ and $-\frac{\pi}{2} \le y_2(2\pi) \le 0$. In particular, the interval $[y_2(2\pi), y_1(2\pi)] \subset [-\frac{\pi}{2}, \frac{\pi}{2}] \subset [y_2(0), y_1(0)]$. Now we can define a Poinccaré map $P : [y_2(0), y_1(0)] \to [y_2(0), y_1(0)]$ by solving the system with $x(0) = 0$, $y(0) = y_0$ to get $P(y_0) = y(2\pi)$. It takes

$$P : [y_2(0), y_1(0)] \to [y_2(2\pi), y_1(2\pi)] \subset [y_2(0), y_1(0)]$$

because solutions can't cross y_1 or y_2. P is continuous by continuous dependence on initial conditions. By the fixed point theorem for continuous self-maps of compact intervals, there is a fixed point $y^* \in [y_2(0), y_1(0)]$ such that $P(y^*) = y^*$ which corresponds to a 2π-periodic solution. By 2π periodicity in y, all $y^* + 2\pi k$ for integers k are also solutions.

A second family of periodic solutions can be found by considering the two solutions $y_1(x)$ and $y_3(x) = y_2(x) + 2\pi$. Because $y_1(\pi) \le \pi \le y_3(\pi)$, we have $y_1(x) \le y_3(x)$ for all $0 \le x \le 2\pi$. This time $[y_1(0), y_3(0)] \subset [\frac{\pi}{2}, \frac{3\pi}{2}] \subset [y_1(2\pi), y_3(2\pi)]$ so we construct a backward Poincaré map

$$Q : [y_1(2\pi), y_3(2\pi)] \to [y_1(0), y_3(0)] \subset [y_1(2\pi), y_3(2\pi)]$$

by using the solution with $x(0) = 2\pi$ so $x(-2\pi) = 0$ and $y(0) = y_{2\pi}$ and setting $Q(y_{2\pi}) = y(-2\pi)$. This gives another continuous map from a compact interval to itself, thus has a fixed point $y^{**} \in [y_1(0), y_3(0)]$ such that $Q(y^{**}) = y^{**}$. This gives another periodic solution family $y^{**}(x) + 2\pi\ell$ for integers ℓ.

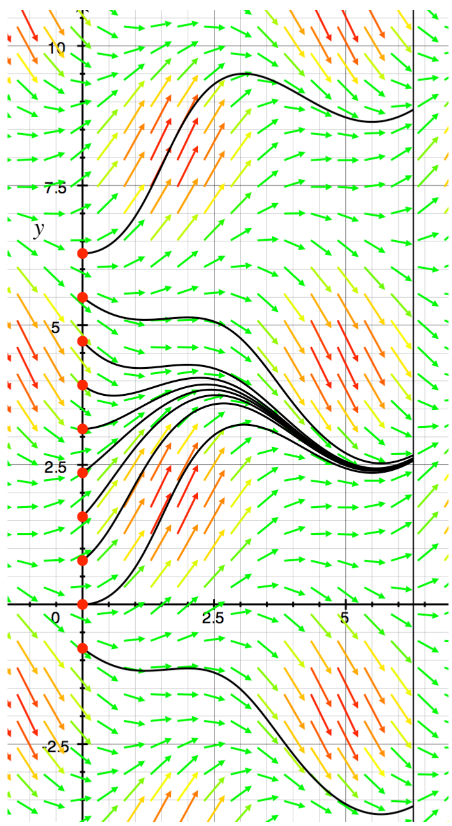

Figure 1: Several trajectories. Periodic solutions have $y^* \approx 2.52$ and $y^{**} \approx 5.18$.

9. *Consider the system of coupled oscillators on the torus with parameters* $\omega_1, \omega_2, K_1, K_2 > 0$

$$\dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1)$$
$$\dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2)$$

Show that the system has no fixed points. Find a conserved quantity for the system. Hint: solve for $\sin(\theta_2 - \theta_1)$ *in two ways. Show that if* $K_1 = K_2$*, the system can be non-*

dimensionalized to

$$\theta_1' = 1 + a\sin(\theta_2 - \theta_1)$$
$$\theta_2' = \omega + a\sin(\theta_1 - \theta_2)$$

Find the winding number ν *analytically. Hint: evaluate the long time averages* $\langle\theta_1' + \theta_2'\rangle$ *and* $\langle\theta_1' - \theta_2'\rangle$*. [Strogatz 8.6.2.]*

$$\nu = \lim_{\tau\to\infty}\frac{\theta_1(\tau)}{\theta_2(\tau)}, \qquad \langle g\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T g(\tau)\,d\tau$$

This was a homework problem. It's included here because noone found a conserved quantity.

Solving $\dot\theta_1 = 0$ and $\dot\theta_2 = 0$ for $\sin\phi$ where $\phi = \theta_1 - \theta_2$ we find

$$\sin(\phi) = \frac{\omega_1}{K_1} = -\frac{\omega_2}{K_2}$$

which has no solution because a negative number can't equal a positive one. The same equation at general points says

$$\sin(\phi) = \frac{\omega_1 - \dot\theta_1}{K_1} = \frac{\dot\theta_2 - \omega_2}{K_2}$$

or

$$\frac{K_2\dot\theta_1 + K_1\dot\theta_2}{K_2\omega_1 + K_1\omega_2} = 1$$

whose solution is

$$\frac{K_2\theta_1(t) + K_1\theta_2(t)}{K_2\omega_1 + K_1\omega_2} = t + t_0$$

where

$$\frac{K_2\theta_1(0) + K_1\theta_2(0)}{K_2\omega_1 + K_1\omega_2} = t_0.$$

The other easily obtained equation is

$$\dot\phi = \dot\theta_1 - \dot\theta_2 = \omega_1 - \omega_2 - (K_1 + K_2)\sin\phi$$

This equation may be explicitly integrated using the $u = \tan(\theta/2)$ substitution. However, we have often discussed the solution but by graphical means. Let us denote its solution from $\phi(0) = \phi_0$ by

$$\phi(t;\,\phi_0)$$

It converges to a fixed point if $|\omega_2 - \omega_1| \le K_1 + K_2$ and grows linearly otherwise. In particular $\phi(t;\,\phi_0)$ is defined for all $-\infty < t < \infty$. We regard (T, P) as new coordinates of the (θ_1, θ_2) plane where

$$T = \frac{K_2\theta_1 + K_1\theta_2}{K_2\omega_1 + K_1\omega_2}, \qquad P = \theta_1 - \theta_2$$

The trajectory starting at (t_0, ϕ_0) of the system in these coordinates is

$$T(t) = t; \qquad P(t) = \phi(t - t_0;\,\phi_0)$$

Such trajectories foliate the whole plane.

The invariant is easily described: we take the P coordinate of the trajectory through the point (θ_1, θ_2) where it crosses the line $T = 0$. It is unchanged for any other point on the trajectory, thus is an invariant of the flow. Note that it is defined on the plane and not on the torus. Analytically it is

$$V(\theta_1, \theta_2) = \phi\left(-\frac{K_2\theta_1 + K_1\theta_2}{K_2\omega_1 + K_1\omega_2};\ \theta_1 - \theta_2\right)$$

To check that it is invariant under the flow, let us use the semigroup property

$$\phi(\tau + s;\ \phi_0) = \phi(\tau;\ \phi(s; t_0, \phi_0))$$

Thus for any point on the integral curve $(\theta_1(s), \theta_2(s))$ starting from $(\theta_1(0), \theta_2(0))$ we have

$$\begin{aligned}
V(\theta_1(s), \theta_2(s)) &= \phi\left(-\frac{K_2\theta_1(s) + K_1\theta_2(s)}{K_2\omega_1 + K_1\omega_2};\ \theta_1(s) - \theta_2(s)\right) \\
&= \phi\left(-s - \frac{K_2\theta_1(0) + K_1\theta_2(0)}{K_2\omega_1 + K_1\omega_2};\ \phi(s;\ \theta_1(0) - \theta_2(0))\right) \\
&= \phi\left(-\frac{K_2\theta_1(0) + K_1\theta_2(0)}{K_2\omega_1 + K_1\omega_2};\ \theta_1(0) - \theta_2(0)\right)
\end{aligned}$$

which doesn't depend on s.

To see the non-dimensionalization if $K = K_1 = K_2$, we divide both equations by ω_1

$$\begin{aligned}
\frac{1}{\omega_1}\dot{\theta}_1 &= 1 + \frac{K}{\omega_1}\sin(\theta_2 - \theta_1) \\
\frac{1}{\omega_1}\dot{\theta}_2 &= \frac{\omega_2}{\omega_1} + \frac{K}{\omega_1}\sin(\theta_1 - \theta_2)
\end{aligned}$$

which does the trick with $a = K/\omega_1$, $\omega = \omega_2/\omega_1$ and $\tau = \omega_1 t$.

To find the winding number,

$$\nu = \lim_{\tau\to\infty}\frac{\theta_1(\tau)}{\theta_2(\tau)} = \frac{\lim_{\tau\to\infty}\dfrac{\theta_1(\tau)+\theta_2(\tau)}{\tau} + \lim_{\tau\to\infty}\dfrac{\theta_1(\tau)-\theta_2(\tau)}{\tau}}{\lim_{\tau\to\infty}\dfrac{\theta_1(\tau)+\theta_2(\tau)}{\tau} - \lim_{\tau\to\infty}\dfrac{\theta_1(\tau)-\theta_2(\tau)}{\tau}}$$

Observe that

$$\theta_1' + \theta_2' = 1 + \omega$$

so that

$$\theta_1(\tau) + \theta_2(\tau) = (1 + \omega)\tau + c_1.$$

Thus

$$\lim_{\tau\to\infty}\frac{\theta_1(\tau)+\theta_2(\tau)}{\tau} = \omega + 1$$

Also observe that

$$\phi' = \theta_1' - \theta_2' = 1 - \omega - 2a\sin(\theta_1 - \theta_2) = 1 - \omega - 2a\sin(\phi) = f(\phi;\ a, \omega)$$

By the Fundamental Theorem of Calculus

$$\frac{\theta_1(\tau) - \theta_2(\tau) - \theta_1(0) + \theta_2(0)}{\tau} = \frac{1}{\tau}\int_0^\tau \phi'(s)\,ds = \frac{1}{\tau}\int_0^\tau 1 - \omega - 2a\sin\phi(s)\,ds$$

In case $|1-\omega| \le 2a$, the solution $\phi(\tau;\phi_0)$ converges to a fixed point. In particular, the vector field $f(\phi(s); a,\omega)$ tends to zero as $\phi(s)$ tends to the fixed point so

$$\lim_{\tau\to\infty} \frac{\theta_1(\tau)-\theta_2(\tau)}{\tau} = \langle 1-\omega-2a\sin\phi(s)\rangle = 0.$$

Thus, in the $|1-\omega| \le 2a$ case there is phase-locking

$$\nu = \frac{1+\omega+0}{1+\omega-0} = 1.$$

In case $|1-\omega| > 2a$, the solution $\phi(s)$ is periodic. It revolves a total angle 2π over one period S, whose duration is (p. 108)

$$S = \left|\int_0^{2\pi} \frac{ds}{d\phi}\, d\phi\right| = \left|\int_0^{2\pi} \frac{1}{1-\omega-2a\sin\phi}\, d\phi\right| = \frac{2\pi}{\sqrt{(1-\omega)^2-4a^2}}$$

Thus $\phi(s) = \operatorname{sgn}(1-\omega)2\pi/S\, s + g(s)$ where g is a bounded (oscillatory) function and the sign gives the direction of the flow. It follows that

$$\lim_{\tau\to\infty} \frac{\theta_1(\tau)-\theta_2(\tau)}{\tau} = \lim_{\tau\to\infty} \frac{\phi(\tau)}{\tau} = \operatorname{sgn}(1-\omega)\frac{2\pi}{S} = \operatorname{sgn}(1-\omega)\sqrt{(1-\omega)^2-4a^2}.$$

In case $|1-\omega| > 2a$,

$$\nu = \frac{1+\omega+\operatorname{sgn}(1-\omega)\sqrt{(1-\omega)^2-4a^2}}{1+\omega-\operatorname{sgn}(1-\omega)\sqrt{(1-\omega)^2-4a^2}}.$$

10. *For the given system in the plane,*

$$\begin{aligned}\dot{x} &= y\\ \dot{x} &= x-2x^3+y(x^2-x^4-y^2)\end{aligned}$$

(a) *Draw the phase portrait.*

(b) *What are the attracting sets and attractors for this system of differential equtions? [Hint: consider $V(x,y)=\frac{1}{2}(x^4-x^2+y^2)$.]*

(c) *Does the attractor have sensitive dependence on initial conditions? [from Clark Robinson,* An Introduction to Dynamical Systems, Continuous and Discrete*, Pearson Prentice Hall, 2004. p.287.]*

The fixed points are $y=0$ and solutions of $x(1-2x^2)=0$ or $x=0,\pm 2^{-1/2}$. The Jacobian is

$$J = \begin{pmatrix} 0 & 1\\ 1-6x^2+2xy-4x^3y & x^2-x^4-3y^2\end{pmatrix}$$

so

$$J(0,0) = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \qquad J\left(0,\pm\frac{1}{\sqrt{2}}\right) = \begin{pmatrix} 0 & 1\\ -2 & \frac{1}{4}\end{pmatrix}$$

which is a saddle and an unstable spiral.

We have

$$\dot{V} = -2y^2V(x,y)$$

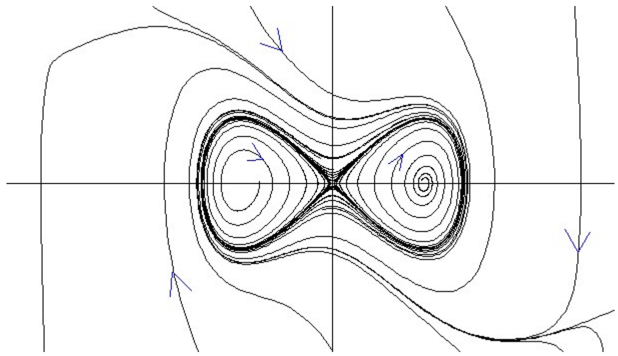

Figure 2: Phase portrait using 3D-XplorMath©.

so that level sets of $V(x, y) = 0$ are solutions. They give two homoclinic orbits $H_\pm$ in the shape of a bow tie on the left and right of the origin. Let $I_\pm$ be the closed insides of the bows. Let Q be the closed outside. Let $\alpha = 2^{-1/2}$. Then all of these sets are invariant for forward and backward flows

$$\mathbf{R}^2, Q, I_+, I_-, I_+ \cup I_-, H_+, H_-, H_+ \cup H_-, \{(0,0)\}, \{(\alpha,0)\}, \{(-\alpha,0)\},$$
$$\{(0,0),(-\alpha,0)\}, \{(0,0),(\alpha,0)\}, \{(-\alpha,0),(\alpha,0)\}, \{(0,0),(-\alpha,0),(\alpha,0)\}$$

Sets like $\{(x, y) : V(x, y) \le \frac{1}{2}\}$ are forward invariant sets because the flow for points with $V > \frac{1}{2}$ decreases V. To be an attractor, it has to attract a neighborhood and be minimal with respect to these properties. Because $V < 0$ inside $I_\pm$, the forward flow is outward away from $(\pm\alpha, 0)$. Thus an open set attracted by $H_+ \cup H_-$ is $U = \mathbf{R}^2 - \{(-\alpha, 0), (\alpha, 0)\}$. Because $V > 0$ outside $I_\pm$, the forward flow is in toward $I_+ \cup I_-$. Thus the following of these attract a neighborhood

$$\mathbf{R}^2, Q, I_+ \cup I_-, H_+ \cup H_-$$

For example, the sets I_+ or H_+ by themselves do not attract points of I_- to the left of but arbitrarily close to the origin. An attractor is minimal with respect to these properties. Thus the only attractor is $H_+ \cup H_-$.

To be chaotic, it must have sensitive dependence on initial conditions. Let us consider an alternative definition of sensitive dependence, due to Robinson.

Definition 1. *A system* $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ *is said to have* sensitive dependence on initial conditions at $\mathbf{x}_0$ *provided there is an* $r > 0$ *such that for every* $\delta > 0$ *there is some* $\mathbf{y}_0$ *with* $|\mathbf{x}_0 - \mathbf{y}_0| < \delta$ *and a time* $\tau > 0$ *such that*

$$|\phi(\tau, \mathbf{y}_0) - \phi(\tau, \mathbf{x} - 0)| \ge r.$$

As the size of the perturbation is taken smaller and smaller, the point $\mathbf{y}_0$ *may change and the time* τ *may change, generally by getting larger. A set* $\mathbf{S}$ *is said to have* sensitive dependence on initial conditions at points in a set $\mathbf{S}$ *provided that it has sensitive dependence on initial conditions for all points* $\mathbf{x}_0 \in \mathbf{S}$*. In general the points* $\mathbf{y}_0$ *can be taken outside the set* $\mathbf{S}$*. However,* $\mathbf{S}$ *is said to have* sensitive dependence on initial conditions when restricted to $\mathbf{S}$ *provided that there is an* $r > 0$ *such that for every* $\delta > 0$ *there is some* $\mathbf{y}_0 \in \mathbf{S}$ *with* $|\mathbf{x}_0 - \mathbf{y}_0| < \delta$ *and a time* $\tau > 0$ *such that*

$$|\phi(\tau, \mathbf{y}_0) - \phi(\tau, \mathbf{x} - 0)| \ge r.$$

An attractor $\mathbf{A}$ is said to be *chaotic* if it has sensitive dependence on initial conditions when restricted to $\mathbf{A}$. Strogatz does not put it this strongly.

Note that the given system exhibits sensitive dependence on initial conditions at points in a set $A = H_+ \cup H_-$. This is because $\mathbf{x}_0 \in A$ converges to the origin, but we can take nearby points $\mathbf{y}_0$ arbitrarily close but outside A which have orbits that will continue around the second bow in the tie. Thus $r = 1$ would work.

However, the given system does not exhibit sensitive dependence on initial conditions at points when restricted to the set A. On this set $H_+ \cup H_-$ where $V = 0$ so

$$\dot{x} = y = \pm x\sqrt{1 - x^2}.$$

Only some of the points in A, those starting near the origin in the unstable manifold at the origin will grow exponentially until they round the bow and then return and die exponentially. For a while the Liapunov exponent is positive for these points. However, the x-coordinate of other points on the stable leg of the bow decreases exponentially, which do not exhibit sensitive dependence. For such the Liapunov exponent is negative.

11. *Consider the Guckenheimer-Williams branched surface cartoon of the Poincaré map for the Lorenz system with standard parameters $\sigma = 10$,$r = 28$ and $b = 8/3$ (see Figure 3). It is a two dimensional model of the attractor which can be thought of as a vertical planar region where the sheets join at the line Σ between the rest points $C_\pm$. On each side of the z-axis, flow from Σ goes downward until it gets ejected outward near the unstable manifold at the origin. It loops around each $C_\pm$ to and returns to Σ. As it sweeps around it spreads out to the whole section Σ.*

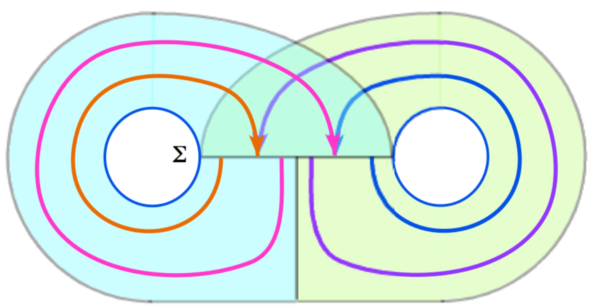

Figure 3: Guckenheimer-Williams branched surface cartoon of the Lorenz attractor.

Notice that the first return of the unstable manifold at the origin is an endpoint of Σ. The first return map $f : \Sigma \to \Sigma$ may be computed and is approximately given in Figure 4.

In this problem we consider a simplified version. $f : [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} 2x, & \text{if } 0 \le x < \frac{1}{2} \\ 2x - 1, & \text{if } 0 \le x < 1, \\ 0, & \text{if } x = 1. \end{cases}$$

This map depicted in Figure 5 is called the "doubling map." Prove that the dynamics given by the doubling map $f : I \to I$ where $I = [0,1]$ exhibits sensitive dependence on initial conditions. In fact, the orbit of any pair of distinct points in I gets at least $1/4$ apart. Show that there is a point $z^* \in I$ so that the orbit of the doubling map is dense in I.

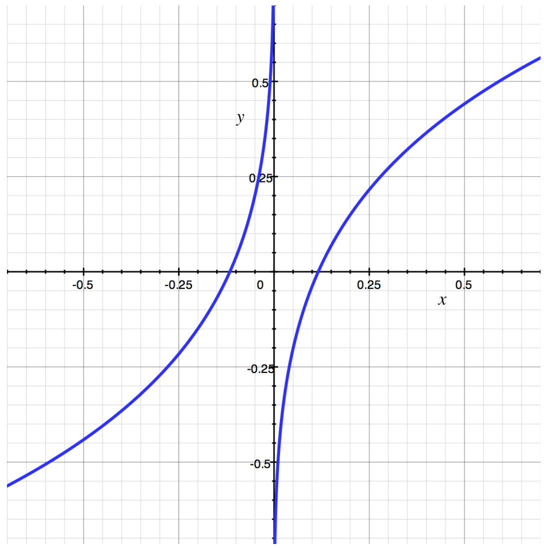

Figure 4: Poincaré Map $f : \Sigma \to \Sigma$ of the branched surface model

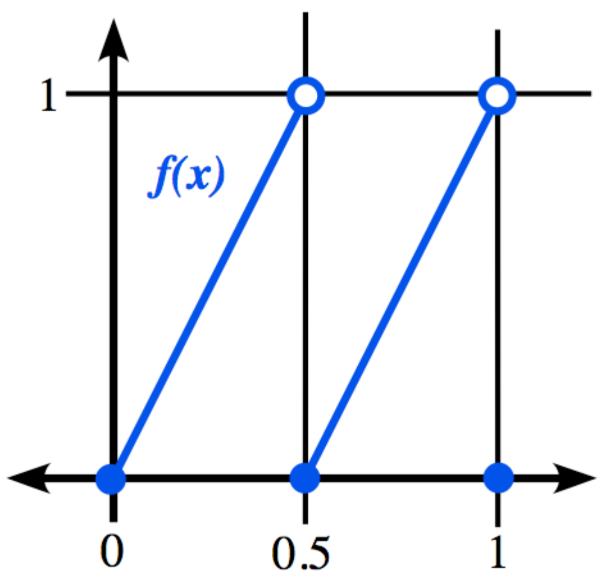

Figure 5: The doubling map.

The key to analyzing the map is to represent numbers $x \in I$ by their binary expansions

$$x = .a_1 a_2 a_3 \ldots = \sum_{i=1}^{\infty} \frac{a_i}{2^i}$$

where each a_i is either 0 or 1. Just as with decimal expansions, some numbers have two different expansions, for example both $.100\ldots$ and $.01111\ldots$ represent the same number $\frac{1}{2}$.

The doubling map $f(x) = 2x \mod 1$ amounts to a shift in the digits.

$$f\left(\sum_{i=1}^{\infty} \frac{a_i}{2^n}\right) = a_1 + \sum_{i=2}^{\infty} \frac{a_i}{2^{i-1}} \mod 1 = \sum_{i=1}^{\infty} \frac{a_{i+1}}{2^i}$$

If x has a repeating binary expansion $a_i = a_{i+p}$ then $x = f^k(x)$ has a periodic orbit with period p.

To see that a number exists whose orbit is dense, we shall write it down. Let $z^* = .0100011011000001010011100101110111\ldots$. The pattern is that the first two digits are 0 then 1, the next eight are 00, 01, 10 and 11, the next 24 run through all three digit numbers from 000 to 111 and so on. Eventually every finite string of ones and zeros is encountered in z^*.

To see that its orbit under f comes arbitrarily close to any number, let $x^* \in I$ be arbitrary, where

$$x^* = \sum_{i=1}^{\infty} \frac{b_i}{2^i}.$$

To show that the orbit of z^* comes within any $\delta > 0$ of x^*, choose k so large that $2^{-k} < \delta$. Now the string $.b_1 b_2 \ldots b_k$ occurs in z^* by construction. Some appropriate iteration then makes $f^p(z^*)$ start with exactly the same first k digits as x^*. For this power

$$\begin{aligned} |f^p(z^*) - x^*| &= \left| \sum_{i=k+1}^{\infty} \frac{a_i^* - b_i}{2^i} \right| \le \sum_{i=k+1}^{\infty} \frac{1}{2^i} \\ &= \frac{1}{2^{k+1}} \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{2^{k+1}} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{1}{2^k} < \delta. \end{aligned}$$

To prove that this map exhibits sensitive dependence on initial conditions when restricted to I, we will show something stronger, namely that f is *expansive*: there is an $r > 0$ such that for any $x_0, y_0 \in I$ such that $x_0 \ne y_0$, there is some iterate p such that

$$|f^p(x_0) - f^p(y_0)| \ge r.$$

The doubling map is expansive with constant $\frac{1}{4}$.

Consider the intervals $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$ and the point $\{1\}$. We show that if $x < y$ and $y - x \le \frac{1}{4}$ then $|f(x) - f(y)| \ge 2|x - y|$. Consider two cases (i) when x and y are in the same interval and (ii) when x and y are in different intervals. Now $f(x) = 2x - K$ where K is 0, 1 or 2, depending on the interval. In case (i), $f(x) - f(y) = 2x - K - 2y + K = 2(x - y)$, so the distance is doubled in this case. In case (ii), the value of K_y for y is larger than the K_x for x. Hence

$$f(x) - f(y) = 2x - K_x - 2y + K_y = (K_y - K_x) - 2(y - x) \ge 1 - 2|y - x|.$$

Now if $|y - x| = y - x \le \frac{1}{4}$ then $1 - 2|y - x|$ is positive and

$$|f(y) - f(x)| = f(y) - f(x) \ge 1 - 2|y - x| \ge \frac{1}{2} \ge 2|y - x|.$$

For any two points $x_0, y_0 \in I$, as long as the distance between $x_j = f^j(x_0)$ and $y_j = f^j(y_0)$ stays less than or equal to $\frac{1}{4}$ for $0 \le j < k$ then

$$|y_k - x_k| \ge 2|y_{k-1} - x_{k-1}| \ge 2^k |y_0 - x_0|.$$

This cannot hold for all $k \ge 0$ so that eventually, $|x_k - y_k| > \frac{1}{4}$. Hence the Liapunov exponent is positive too. [Argument taken from from Clark Robinson, An Introduction to Dynamical Systems, Continuous and Discrete, Pearson Prentice Hall, 2004, sections 7.3, 10.2, 10.4.]

12. *Show that the map has a unique fixed point. Is it stable?*

$$x_{n+1} = \exp(-x_n^2)$$

Observe that $f(x) = \exp(-x^2)$ is bounded $0 \le f(x) \le 1$ for all x. Thus the map sends the interval $f : [0,1] \to [0,1]$ to itself. Because f is continuous, the fixed point theorem for continuous maps on compact intervals guarantees the existence of a fixed point $x^* \in [0,1]$ such that $f(x^*) = x^*$.

If there are any other fixed points, they would have to be in the interval $[0,1]$. To argue that the fixed point is unique, we need to show that x^* is the only zero of the function $g(x) = x - f(x) = x - \exp(-x^2)$. For $x \in [0,1]$ the derivative is strictly positive

$$g'(x) = 1 + 2\exp(-x^2)x \ge 1 + 0 = 1$$

which means $g(x) < 0$ if $x < x^*$ and $g(x) > 0$ if $x > x^*$ so x^* is the unique zero. One can use the mean value theorem to see this: for example if $0 \le x < x^*$ then

$$0 - g(x) = g(x^*) - g(x) = g'(c)(x^* - x) > 0$$

where $x < c < x^*$.

To see if x^* is stable, from calculus we observe that the derivative

$$f'(x) = -2xe^{-x^2}$$

is minimized when $x = \frac{1}{\sqrt{2}}$ so at the fixed point $x^* = f(x^*)$

$$0 \ge f'(x^*) \ge f'\left(\frac{1}{\sqrt{2}}\right) = -\sqrt{2}\exp\left(-\frac{1}{2}\right) = -\sqrt{\frac{2}{e}} > -1$$

thus $|f'(x^*)| < 1$ and x^* is stable.