Math 5470 § 1. **First Midterm Exam** Name: Solutions
Treibergs Feb. 8, 2024.

1. *Consider the equation on the line. Sketch the phase portrait. Find the rest points and determine their stability. Find the potential function $V(x)$. Sketch the potential function use it to check the stability of your rest points from (a) again.*

$$\dot{x} = -4x^3 + 4x$$

Factoring

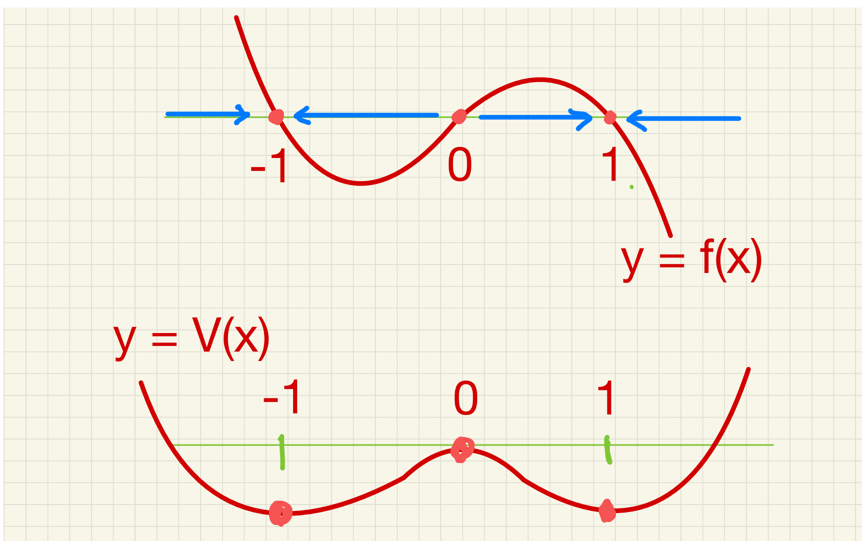
$$f(x) = -4x^3 + 4x = 4x(1 - x^2) = 4x(1 - x)(1 + x)$$

so that the rest points are $0, 1, -1$. $f(x) > 0$ for $x < -1$ or $0 < x < 1$ and negative for $-1 < x < 0$ or $1 < x$. Thus $-1, 1$ are stable rest points and 0 is unstable.

The potential satisfies $V'(x) = -f(x)$ so, up to an additive constant,

$$V(x) = -\int_0^x f(z)\,dz = x^4 - 2x^2$$

This is a "W"-shaped potential with minima at $x = \pm 1$ which are stable and a relative max at $x = 0$ which is unstable.



2. *Determine whether the given equilibrium point for the given system is Attractive, is Liapunov Stable, or is Not Stable. Give a brief explanation.*

(a) $\theta = 0$ *for* $\dot{\theta} = 1 - \cos\theta$.

Attractive.

$f(\theta) = 0$ only at $\theta = 0$ and is positive elsewhere else on the circle. Thus either $\theta(0) = 0$ so flow stays at rest point or $\theta(0) > 0$ and flow advances until it returns to the rest point. Thus the rest point is attractive. It is not Liapunov stable because for small neighborhoods, $U = (-\epsilon, \epsilon)$ of zero where $0 < \epsilon < \pi$, starting at $0 < \theta(0) < \epsilon$, the flow exits U before it returns to zero.

(b) $\theta = -\dfrac{\pi}{2}$ *for* $\dot{\theta} = \cos^3\theta$

Not Stable.

$\cos^3\theta$ is negative for $-\frac{3\pi}{2} < \theta < -\frac{\pi}{2}$ and positive for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ thus flow is away from the point $\theta = -\frac{\pi}{2}$ making it unstable.

(c) $(x, y) = (0, 0)$ *for* $\begin{cases} \dot{x} = -3x + 2y \\ \dot{y} = -4x + y \end{cases}$

Both Attractive and Liapunov Stable. so Asymptotically Stable.
The trace of $\begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix}$ is $\tau = -2$ and determinant $\Delta = 5$ so $1 = \frac{1}{4}\tau^2 < \Delta = 5$. From the trace-determinant plane, or by computing the eigenvalues, the roots of $\lambda^2 - \tau\lambda + \Delta = 0$ which are

$$\lambda = \frac{-2 \pm \sqrt{4 - 4 \cdot 5}}{2} = -1 \pm 2i,$$

the trajectories are stable spirals, making the origin both attractive and Liapunov stable.

(d) $(x, y) = (0, 0)$ *for* $\begin{cases} \dot{x} = 2x - 4y \\ \dot{y} = 3x - 6y \end{cases}$

Liapunov Stable.
The trace of $\begin{pmatrix} 2 & -4 \\ 3 & -6 \end{pmatrix}$ is $\tau = -4$ and determinant $\Delta = 0$. From the trace-determinant plane, or by computing the eigenvalues, the roots of $\lambda^2 - \tau\lambda + \Delta = 0$ we have

$$\lambda = \frac{-4 \pm \sqrt{16 - 0}}{2} = -4, \quad 0.$$

Thus the trajectories form a stable comb. There is a line of rest points through the origin ($x = 2y$) and the flow is toward this line along paths (parallel to $(2, 3)$). Points stay in a neighborhood of the origin if they begin close enough to it, but not all nearby points tend to the origin so it is not attractive.

3. *Let $\theta(t)$ be the phase in the circle of a firefly's flashing rhythm, where $\theta(t) = 0$ corresponds to the instant when the flash is emitted. Assume that the firefly's natural frequency is ω. If it senses a stimulus $\psi(t)$ at frequency Ω, then it tries to adjust according to the system. Show that for Ω close enough to ω, the firefly manages to synchronize with the stimulus, but if Ω is sufficiently different, it fails to synchronize. How close is "close enough"?*

$$\dot{\psi} = \Omega$$
$$\dot{\theta} = \omega + \sin(\psi - \theta)$$

Let $\varphi = \psi - \theta$ Then φ satisfies

$$\dot{\varphi} = \dot{\psi} - \dot{\theta} = \Omega - \omega - \sin(\psi - \theta) = (\Omega - \omega) - \sin\varphi.$$

The firefly synchronizes to the flashing if there is a stable fixed point φ_- and then

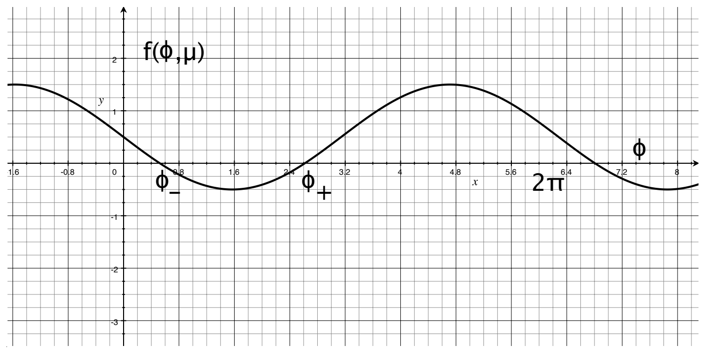
$$\psi(t) - \omega(t) = \varphi(t) \to \varphi_- \qquad \text{as } t \to \infty.$$

In other words, the firefly settles to the same frequency as the flashing except with a time delay of θ_-.

There is a stable fixed point if and only if $\mu = \Omega - \omega$ satisfies $|\mu| < 1$. To see this, the fixed points are the zeros of

$$f(\varphi, \mu) = \mu - \sin\varphi$$

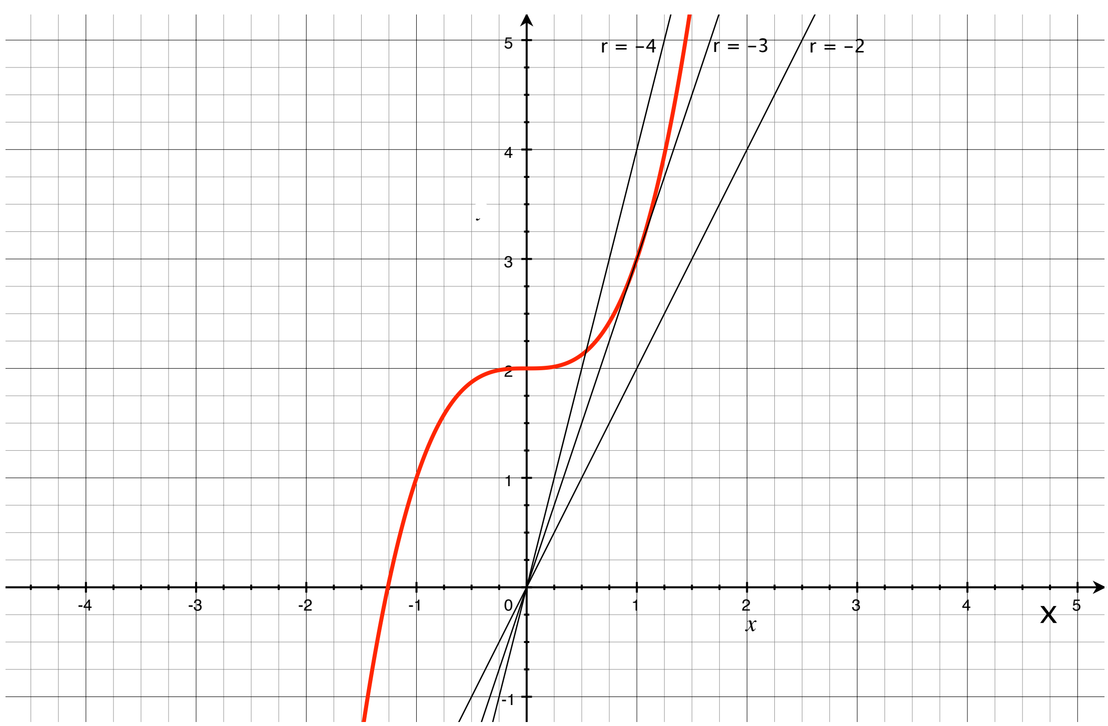
which occur at $\varphi_-, \varphi_+ \in \sin^{-1}(\mu)$. For $0 \le \mu < 1$ as in the figure, we have $0 \le \varphi_- < \frac{\pi}{2} < \varphi_+ \le \pi$ and for $-1 < \mu < 0$ we have $\pi < \varphi_+ < \frac{3\pi}{2} < \varphi_- < 2\pi$. In both cases $f > 0$ for nearby $\varphi < \varphi_-$ and $f < 0$ for $\varphi_- < \varphi$, showing that φ_- is a stable rest point.



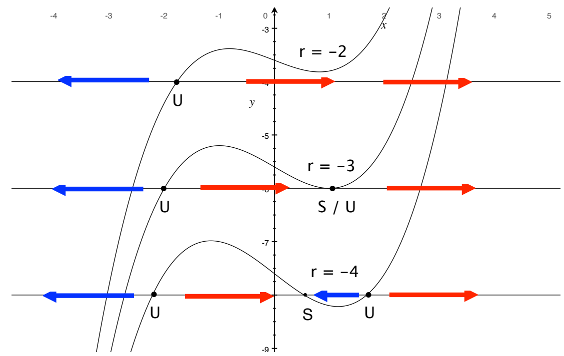
4. *Sketch the qualitatively different vector fields that occur as r is varied. Find and classify the bifurcation points. Sketch the bifurcation diagram.*

$$\dot{x} = 2 + rx + x^3 = f(x, r)$$

We look at $y = 2+x^3$ and $y = -rx$ for $r = -2, -3, -4$ and see that the equation $2+x^3 = -rx$ has one, two and three intersection points corresponding to the zeros of $f(x,r) = 0$. f is positive when $y = 2 + x^3$ is above $y = -rx$ and negative when below.



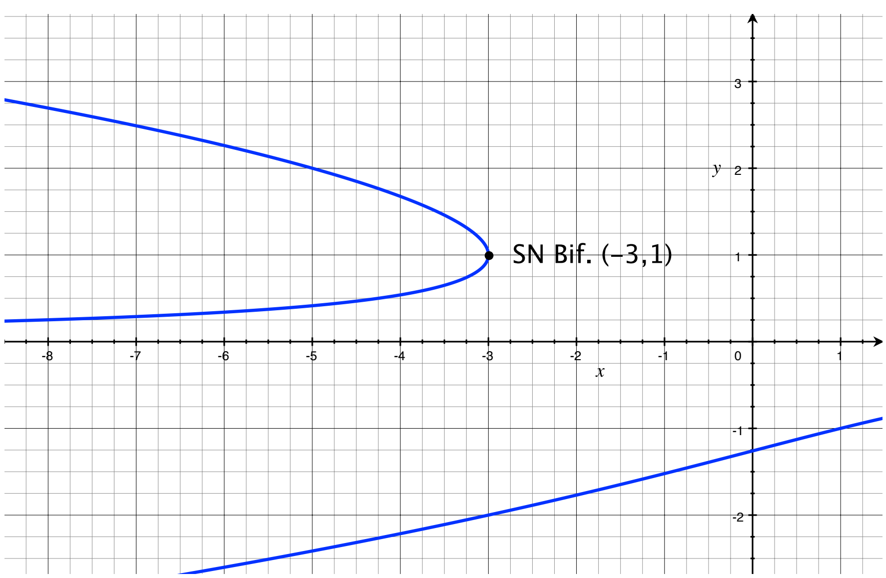
Thus plotting $f(x,r)$ for several r values shows the stability of the rest points. At $r=-3$ and $x=1$ a saddle-node bifurcation point appears as r decreases through $r=-3$.



For $r>-3$ there is only one negative unstable rest point. For $r=-3$ a second rest point appears at $x=1$. As r decreases from $r=-3$ the rest point splits into a stable rest point below $x=1$ and an unstable one abopve $x=1$, making three rest points in the $r<-3$ regime.

The bifurcation diagram is the locus of $f(x,r)=0$. This is most easily plotted by solving for r

$$r=-\frac{2}{x}-x^2.$$



5. *The predation on a population $P(N)$ is very fast and a model of the prey $N(t)$ satisfies an ODE with small $0 < \varepsilon$ and with R, K, P and A positive constants.*

$$\frac{dN}{dt} = RN\left(1 - \frac{N}{K}\right) - P\left\{1 - \exp\left(-\frac{N^2}{\epsilon A^2}\right)\right\}$$

What are the dimensions of R, K, P and A? Find dimensionless quantities x, τ, and parameters r and q so that the equation can be put into the dimensionless form

$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{q}\right) - \left\{1 - \exp\left(-\frac{x^2}{\epsilon}\right)\right\}$$

Show that the system can have three rest points if the parameteers r and q lie in the region approximately given by $rq > 4$. What is hysteresis? Could this system exhibit hysteresis?

$N(t)$ is population, which has the same dimension as A and K. The left side has the dimension (pop.)(time)$^{-1}$, thus R has dimension (time)$^{-1}$ and P has dimension (pop.)(time)$^{-1}$.

Writing in terms of $x = \frac{N}{A}$, dimensionless population, the equation becomes

$$A\frac{dx}{dt} = RAx\left(1 - \frac{Ax}{K}\right) - P\left\{1 - \exp\left(-\frac{x^2}{\epsilon}\right)\right\}$$

Dividing through by P,

$$\frac{dt}{d\tau}\frac{dx}{dt} = \frac{A}{P}\frac{dx}{dt} = \frac{RA}{P}x\left(1 - \frac{Ax}{K}\right) - \left\{1 - \exp\left(-\frac{x^2}{\epsilon}\right)\right\}$$

This suggests that we set $\tau = \frac{P}{A}t$, dimensionless time and $r = \frac{RA}{P}$ and $q = \frac{K}{A}$ dimensionless constants, yielding

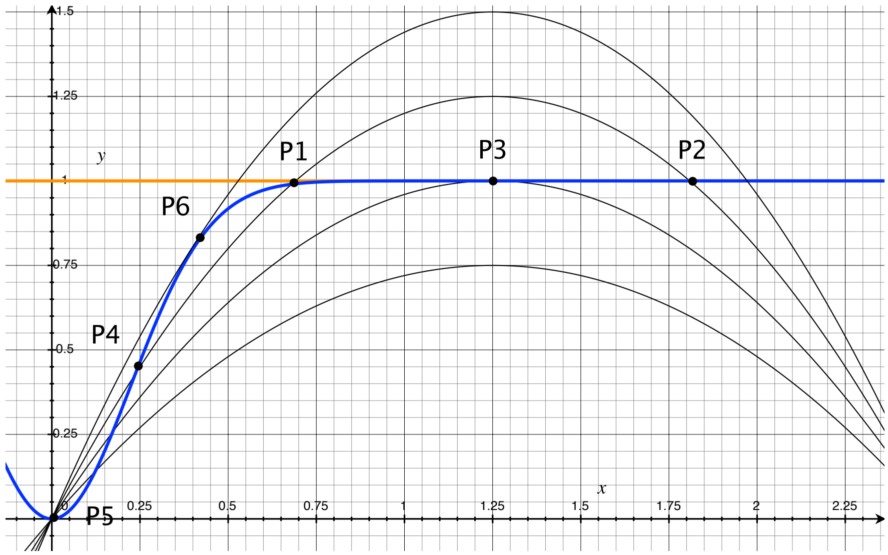
$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{q}\right) - \left\{1 - \exp\left(-\frac{x^2}{\epsilon}\right)\right\}.$$

Fixing q, we may consider what happens as r increases from zero. The rest points are the intersections of the left and right sides of

$$rx\left(1 - \frac{x}{q}\right) = \left\{1 - \exp\left(-\frac{x^2}{\epsilon}\right)\right\}.$$

For small ϵ, the right side quickly increases from zero to $y = 1$.

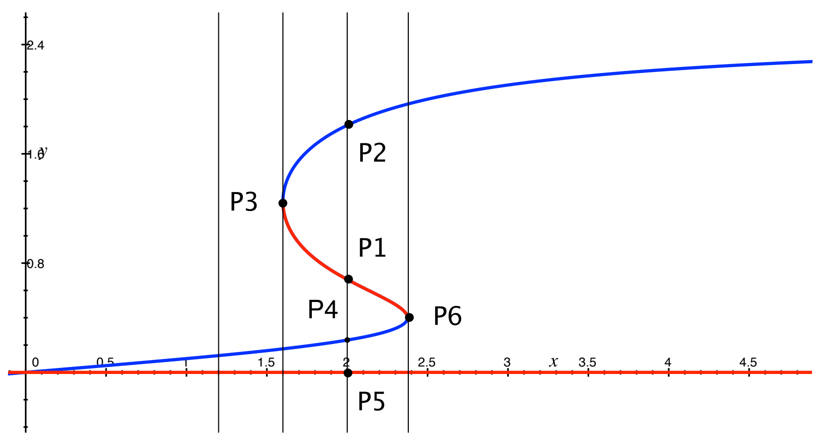
The maximum of the left side occurs at the point $(\frac{q}{2}, \frac{rq}{4})$. Thus when $rq > 4$ the maximum exceeds one and the parabola is above the right side. The plot is for $q = 2.5$ and $\epsilon = .1$ and for values $r = 1.2$, 1.6, 2, 2.38. The blue curve is the right side. At $r \approx 1.6$, there is a saddle-node bifurcation at P_3. As r increases this splits into unstable and stable fixed points P_1 and P_2, pictured at $r = 2$. As r increases, the stable and unstable rest points P_4 and P_1 coincide at another saddle node bifurcation at P_6 at $r = 2.38$.



Solving for r at the rest points gives

$$r = \frac{1 - \exp\left(-\frac{x^2}{\epsilon}\right)}{x\left(1 - \frac{x}{q}\right)}.$$

whose plot together with $x = 0$ is the bifurcation diagram. The blue curves of rest points are stable and the red unstable. In particular, there are four rest points when $1.6 < r < 2.38$.



Hysteresis is the lack of reversibility of the solution as the parameter is altered. This equation exhibits hysteresis. For example, starting at the stable point P_2 at $r = 2$, we decrease r until it dips below the bifurcation value $r = 1.6$. The stable point gets dragged along the upper curve until it passes the bifurcation point at P_3 and then jumps to the only remaining stable rest point on the lower blue line. Then when the parameter is increased back to $r = 2$ we're at the rest point P_4 on the lower stable branch instead of where we started at P_2. Continuing to raise r past $r = 2.38$, the rest point passes another bifurcation point at P_6 where it jumps back to the only stable point, which is on the upper blue line. Decreasing the parameter down to $r = 2$ returns the rest point to P_2