Homework for Math 5470 §002, Spring 2024

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April 9, 2024

Our text is by Steven H. Strogatz, *Nonlinear Dynamics and Chaos*, 2nd ed. Please read the relevant sections in the text as well as any cited reference. Assignments are due the following Tuesday, or on Apr. 23, whichever comes first.

Your written work reflects your professionalism. Make answers complete and self contained. This means that you should copy or paraphrase each question, write in complete sentences, provide adequate explanation to help the reader understand the structure of your argument, be thorough in the details, state any theorem that you use and proofread your answer.

Homework from Thursday and Tuesday will be due Tuesday. Late homework that is up to one week late will receive half credit. Homework that is more than one week late will receive no credit at all. Homework that is placed in my mailbox in JWB 228 before 4:00 pm Tuesday afternoon will be considered to be on time.

Please hand in problems A on Tuesday, January 16.

A. Exercises from Chapter 2 of the text

`2.2[3,6,8,12], 2.3[2,3,4], 2.4[2]`

2.2.3 Analyze graphically: sketch the vector field, find the fixed points, classify their stability, sketch graph for various initial conditions. Find the analytic solution if it's easy.

$$\dot{x} = x - x^3$$

2.2.6 Same for $\dot{x} = 1 - 2\cos x$

2.2.8 Find an $\dot{x} = f(x)$ whose phase portrait is

$$\longrightarrow -1 \longrightarrow 0 \longleftarrow\!\!\!\longleftarrow 2 \longrightarrow$$

2.2.12 Suppose that the resistor $I = V/R$ in the RC circuit Example 2.2.2 is replaced by $I = g(V)$ which is an increasing function with $g(0) = 0$, for example $g(V) = r\,\mathrm{Atn}(V)$ for some $r > 0$. Redo Example 2.2.2 in this case. Derive the circuit equations, find the fixed points and analyze their stability. What qualitative effect does this nonlinearity introduce (if any)?

2.3.2 Autocatalysis for the concentration $x = [X]$ is given by the equation

$$\dot{x} = k_1 x - k_2 x^2$$

where k_i are positive reaction rate constants. Find all fixed points and classify their stability. Sketch the graph of $x(t)$ for various initial values.

2.3.3 Growth of a tumor may be modeled by a Gompertz law

$$\dot{N} = -aN \log(bN)$$

where $N(t)$ is proportional to number of cells in the tumor and $a, b > 0$ are constants. Interpret a and b biologically. Sketch the vector field and graph $N(t)$ for various initial values.

2.3.4 For certain organisms, the effectve growth rate $\dot{N}/N$ is largest at intermediate N. This is called the *Allee Effect.* Show that

$$\frac{\dot{N}}{N} = r - a(N-b)^2$$

provides an example of the Allee Effect, if a, b and r satisfy certain constraints to be determined. Find the fixed points and classify their stability. Sketch the solutions for various initial values. Compare solutions $N(t)$ to those of the Logistic Equation. What are the qualitative differences (if any)?

2.4.2 Use linearized stability analysis to classify the fixed point of $\dot{x} = x(x-1)(x-2)$.

Please hand in problems B on Tuesday, January 23.

B. Exercises from Chapter 2 – 3 of the text

`2.6[2], 3.1[4], 3.2[4,6], 3.3[1,2]`

2.6.2 Here is an analytic proof that periodic solutions are impossible for vector fields on the line. Suppose on the contrary that $x(t)$ is a nontrivial periodic solution, *i.e.,* $x(t) = x(t+T)$ for some $T > 0$, and $x(t) \neq x(s)$ for all $t < s < t+T$. Derive a contradiction by considering

$$\int_t^{t+T} f(x)\frac{dx}{dt}\,dt.$$

3.1.4 Sketch the qualitatively different vector fields as r is varied. Show that a saddle-node bifurcation occurs for a particular value of r, to be determined. Sketch the bifurcation diagram for various fixed points x^* versus r.

$$\dot{x} = r + \frac{1}{2}x - \frac{x}{1+x}$$

3.2.4 Sketch the qualitatively different vector fields as r is varied. Show that a transcritical bifurcation occurs for a particular value of r, to be determined. Sketch the bifurcation diagram for various fixed points x^* versus r.

$$\dot{x} = x(r - e^x)$$

3.2.6 Consider $\dot{X} = RX - X^2 + aX^3 + \mathbf{O}(X^4)$ where $R \neq 0$. We seek a new variable x so that the system transforms to $\dot{x} = Rx - x^2 + \mathbf{O}(X^4)$. Put $x = X + bX^3 + \mathbf{O}(X^4)$, where b will be chosen later to eliminate the cubic term. Rewrite the equation in terms of x using the steps:

(a) Show that this can be inverted to yield $X = x + cx^3 + \mathbf{O}(x^4)$ and solve for c.

(b) Write $\dot{x} = \dot{X} + 3bX^2\dot{X} + \mathbf{O}(X^4)$ and substitute X and $\dot{X}$ on the right side so that everything depends only on x. Multiply the resulting expansion and collect terms to obtain $\dot{x} = Rx - x^2 + kx^3 + \mathbf{O}(x^4)$ where k depends on a, b and R.

(c) Choose b so that $k = 0$.

(d) Is it really necessary to assume $R \neq 0$? Explain.

3.3.1 The laser model of Milonni and Eberly is

$$\dot{n} = GnN - kn$$
$$\dot{N} = -GnN - fN + p$$

where constants $G, k, f > 0$ and p may have either sign.

(a) Assume we can approximate $\dot{N} \approx 0$. Derive a first order equation for n.

(b) Show that $n^* = 0$ becomes stable for $p > p_c$, where p_c is to be determined.

(c) What type of bifurcation occurs at p_c?

(d) For what range of parameters is it valid to make the approximation used in (a)?

3.3.2 The (slightly corrected from Strogatz*) laser model of Maxwell-Bloch (the Haken-Lorenz equations) is

$$\dot{E} = \kappa(P - E)$$
$$\dot{P} = \gamma_1(ED - P)$$
$$\dot{D} = \gamma_2(\lambda + 1 - D - EP)$$

where E is electric field, P is mean polarization and D is population inversion, and where constants $\kappa, \gamma_1, \gamma_2 > 0$ and λ may have either sign. In practical lasers, $\gamma_1, \gamma_2 \gg \kappa$ so D and P relax rapidly to steady values and may be eliminated as follows.

(a) Assume we can approximate $\dot{P} \approx 0$ and $\dot{D} \approx 0$. Express P and D in terms of E to derive an ODE for E.

(b) Find the fixed points in the equation for E

(c) Draw the bifurcation diagram of E^* versus λ. Distinguish stable and unstable branches.

(*) See for example, Roldan and Prati, "Semiclassical Theory of Amplification and Lasing," p.15, `http://arxiv.org/pdf/quant-ph/0605084.pdf`

Please hand in problems C on Tuesday, Jan. 30.

C. Exercises from Chapter 3 of the text

`3.4[15], 3.5[1, 2, 3, 7], 3.6[3], 3.7[6]`

3.4.15 Consider the potential $V(x)$ for the system

$$\dot{x} = rx + x^3 - x^5$$

Calculate the critical r_c defined by the condition that V has three equally deep wells, *i.e.,* the values of V at the three local minima are equal.

3.5.1 Consider the bead on the rotating hoop discussed in Section 3.5. Explain in physical terms why the bead can't have an quilibrium position with $\phi > \frac{\pi}{2}$.

3.5.2 Do the linear stability analysis at all fixed points for Equation (3.5.7) and confirm that Figure 3.5.6 is correct.

3.5.3 Show that Equation (3.5.7) reduces to

$$\frac{d\phi}{d\tau} = A\phi - B\phi^3 + \mathbf{O}(\phi^5)$$

near $\phi = 0$. Find A and B.

3.5.7 Consider the logistic equation

$$\dot{N} = rN\left(1 - \frac{N}{K}\right), \qquad N(0) = N_0.$$

(a) The equation has three dimennsional parameters r, K and N_0. Find the dimensions of each of these parameters.

(b) Show that the system can be written in dimensionless form

$$\frac{dx}{d\tau} = x(1-x), \qquad x(0) = x_0.$$

3.6.3 Consider the system

$$\dot{x} = rx + ax^2 - x^3$$

where a and r are real constants. When $a = 0$ we have the normal form of the supercritical pitchfork.

(a) For each a there is a bifurcation diagram of x^* versus r. As a varies, these diagrams undergo qualitative changes. Sketch all the qualitatively different diagrams for various a.

(b) Summarize your results by plotting in the regions in the (r, a) plane that correspond to qualitatively different vector fields. Bifurcations occur at the boundary of these regions. Identify the type of bifurcations that occur.

3.7.6 Kermack and McKendrick's model for an epidemic is

$$\begin{aligned}
\dot{x} &= -kxy \\
\dot{y} &= kxy - \ell y \\
\dot{z} &= \ell y
\end{aligned}$$

where x, y and z are the number of healthy people, sick people, and dead people, resp., and k and ℓ are positive constants. It ignores slower changes such as births, emigration and deaths due to other causes.

(a) Show that $x + y + z = N$, where N is a constant.

(b) Use the $\dot{x}$ and $\dot{y}$ equations to show that

$$x(t) = x_0 \exp\left(-\frac{k}{\ell}z(t)\right), \qquad \text{where } x_0 = x(0).$$

(c) Show that z satisfies the first order equation

$$\dot{z} = \ell\left[N - z - x_0 \exp\left(-\frac{k}{\ell}z(t)\right)\right].$$

(d) Show this equation may be nondimensionalized to

$$\frac{du}{d\tau} = a - bu - e^{-u}$$

(e) Show $a \geq 1$ and $b > 0$.

(f) Determine the number of fixed points u^* and classify their stability.

(g) Show that the maximum of $\dot{u}(t)$ occurs at the same time as the maximum of $\dot{z}(t)$ and $y(t)$. This time is called the peak of the epidemic and is denoted t_p.

(h) Show that if $b < 1$ then $\dot{u}(t)$ is increasing at $t = 0$ and reaches its maximum at some time $t_p > 0$. Thus things get worse before they get better. Show that $\dot{u}(t)$ eventually decreases to zero. In this situation we say that an epidemic occurs.

(i) On the other hand, show that $t_p = 0$ if $b > 1$. So no epidemic occurs.

(j) $b = 1$ is the threshold condition far the epidemic to occur. Can you give a biological interpretation?

(k) How would you revise the model to make it more appropriate for AIDS? Which assumptions need revising?

Please hand in problems D on Tuesday, Feb. 6.

D. Exercises from Sec. 4.1 – 5.1 of the text

`4.1[5], 4.2[2], 4.3[4], 4.5[3], 4.6[3], 5.1[2]`

4.1.5 Find and classify all the fixed points and sketch the phase portrait on the circle.

$$\dot{\theta} = \sin\theta + \cos\theta$$

4.2.2 Graph $x(t) = \sin 8t + \sin 9t$ for $-20 < t < 20$. You should find that the amplitude is modulated.

i. What is the period of the amplitude modulation?

ii. Solve the problem analytically, using a trig identity that converts sums of sines and cosines to products of sines and cosines.

4.3.4 Draw the phase portrait as a function of the control parameter μ. Classify the bifurcations that occur as μ varies and find all the bifurcation values of μ.

$$\dot{\theta} = \frac{\sin\theta}{\mu + \cos\theta}$$

4.5.3 In an excitable system, as in a neuron, a small signal will cause little response, but above a threshold level the signal will produce a large spike before returning to rest. Roughly speaking, excitable systems have two properties: (1) it has a unique, globally attracting rest state; and (2) a large stimulus can send the system on a long excursion through phase space before it returns to the resting state. Consider the caricature equation

$$\dot{\theta} = \mu + \sin\theta$$

where μ is slightly less than 1.

i. Show that the system satisfies the two properties. What objects plays the role of "rest state" and "threshold"?

ii. Let $V(t) = \cos\theta(t)$. Sketch $V(t)$ for various initial conditions. ($V(t)$ is analogous to a neuron's membrane potential and initial conditions correspond to different perturbations from the rest state.)

4.6.3 As in §2.7, imagine a particle sliding down a suitable potential.

i. Find a potential function corresponding to Equation (4.6.7): $\phi' = I/I_c - \sin\phi$. Show that it is not a single valued function on the circle.

ii. Graph the potential, called the *washboard potential*, as a function of ϕ for various values of I/I_c. Here ϕ is to be regarded as a real number, not an angle.

iii. What is the effect of increasing I?

5.1.2 Consider the system for constant $a < -1$

$$\dot{x} = ax$$
$$\dot{y} = -y$$

Sketch some trajectories. Show that all trajectories become parallel to the y-direction as $t \to \infty$, and parallel to the x-direction as $t \to -\infty$. (Hint: Examine the slope $dy/dx = \dot{y}/\dot{x}$)

Please hand in problems E on Tuesday, Feb. 12.

E. Exercises from Sec. 5.1 – 5.3 of the text

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5.1[9, 10], 5.2[2, 6], 5.3[4]
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5.1.9 Consider the system $\dot{x} = -y$, $\dot{y} = -x$.

i. Sketch the vector field.

ii. Show that trajectories are hyperbolas of the form $x^2 - y^2 = C$.
[Hint: show $x\dot{x} - y\dot{y} = 0$ and integrate.]

iii. The origin is a saddle point. Find equations for its stable and unstable manifolds.

iv. The system can be solved as follows. Introduce new variables $u = x + y$ and $v = x - y$. Rewrite the system in terms of u and v. Solve starting from an arbitrary initial condition (u_0, v_0).

v. What are the equations of the stable and unstable manifolds in terms of u and v?

vi. Using the answer to (d), write the general solution for $x(t)$ and $y(t)$ starting from an initial condition (x_0, y_0).

5.1.10 Here are the official definitions of different types of stability. Consider a fixed point x^* of the system $\dot{x} = f(x)$.
We say that x^* is **attracting** if there is a $\delta > 0$ such that $\lim_{t\to\infty} x(t) = x^*$ whenever $\|x(0) - x^*\| < \delta$. In other words, any trajectory that starts within a distance δ of x^* is guaranteed to converge to x^* *eventually.* Trajectories that start nearby are allowed to stray from x^* in the short run but must converge to x^* in the long run.
In contrast, we say that x^* is **Lyapunov Stable** if for each $\epsilon > 0$ there is a $\delta > 0$ such that $\|x(t) - x^*\| < \epsilon$ whenever $t \geq 0$ and $\|x(0) - x^*\| < \delta$. Thus trajectories that start within δ of x^* stay within ϵ of x^* for all positive time.
Finally, we say x^* is **asymptotically stable** if it is both attracting and Liapunov stable.
For each of the following systems, determine whether the origin is attracting, Liapunov stable, asymptotically stable or none of these. Explain your deductions.

$$(a)\ \begin{cases}\dot{x} = y \\ \dot{y} = -4x\end{cases} \quad (b)\ \begin{cases}\dot{x} = 2y \\ \dot{y} = x\end{cases} \quad (c)\ \begin{cases}\dot{x} = 0 \\ \dot{y} = x\end{cases}$$
$$(d)\ \begin{cases}\dot{x} = 0 \\ \dot{y} = -y\end{cases} \quad (e)\ \begin{cases}\dot{x} = -x \\ \dot{y} = -5y\end{cases} \quad (f)\ \begin{cases}\dot{x} = x \\ \dot{y} = y\end{cases}$$

5.2.2 Consider the system $\dot{x} = x - y$, $\dot{y} = x + y$.

i. Sketch the vector field.

ii. Show that A has the eigenvalues $\lambda, \bar{\lambda} = 1 \pm i$ with corresponding eigenvectors $\mathbf{x}, \bar{\mathbf{x}} = \binom{\pm i}{1}$.

iii. The general solution is $\mathbf{z}(t) = c_1 e^{\lambda t}\mathbf{x} + c_2 e^{\bar{\lambda} t}\bar{\mathbf{x}}$. Write a general solution in terms of purely real functions. Solving for the unknown constants, write the general solution for $x(t)$ and $y(t)$ starting from the initial condition (x_0, y_0).

5.2.6 Plot the phase portrait and classify the fixed point in the following linear system. If the eigenvectors are real, indicate them in your sketch.

$$\begin{aligned} \dot{x} &= -3x + 2y \\ \dot{y} &= x - 2y \end{aligned}$$

5.3.4 Predict the course of the love affair depending on signs and relative sizes of the constants a and b. Suppose Romeo and Juliet are opposites. Analyze

$$\begin{aligned} \dot{R} &= aR + bJ \\ \dot{J} &= -bR - aJ \end{aligned}$$

Please hand in problems F on Tuesday, February 20.

F. Exercises from Sec. 6.3 – 6.8 of the text

`6.3[4, 13], 6.4[2], 6.5[9, 10, 19], 6.6[5]`

6.3.4 Find and classify the rest points. Sketch the neighboring trajectories and fill in the rest of the phase portrait.

$$\begin{aligned} \dot{x} &= y + x - x^3 \\ \dot{y} &= -y \end{aligned}$$

6.3.13 Show that the origin is a spiral although the it's a center for the linearization.

$$\begin{aligned} \dot{x} &= -y - x^3 \\ \dot{y} &= x \end{aligned}$$

6.4.2 Let $x, y \geq 0$ be populations of rabbits and sheep. Find the fixed points. Investigate their stability. Draw the null clines and sketch the phase portrait. Indicate the basin of attraction for any stable fixed point.

$$\begin{aligned} \dot{x} &= x(3 - 2x - y) \\ \dot{y} &= y(2 - x - y) \end{aligned}$$

6.5.9 A *Hamiltonian System* in the system in the plane for which there exists a smooth real function $H(p, q)$ such that

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned}$$

Show that H is a conserved quantity so that trajectories lie on contour lines $H(p, q) = C$.

6.5.10 A particle moves in the plane governed by inverse square force whose Hamiltonian is

$$H(p, r) = \frac{p^2}{2} + \frac{h^2}{2r^2} - \frac{k}{r}$$

where $r > 0$ is the distance to the origin and p is the radial momentum. The parameters h and k are the angular momentum and force constant, resp.

i. Suppose $k > 0$ corresponding to an attracting force of gravity. Sketch the phase portrait in the (r, p) plane. [Hint: graph the effective potential $V(r) = h^2/2r^2 - k/r$ and look for intersections with horizontal lines of height E. Use this to sketch contour curves $H(p, r) = E$ for various positive and negative E.

ii. Show that the trajectories are closed if $-k^2/2h^2 < E < 0$. What happens if $E > 0$?

iii. If $k < 0$ (as in electric repulsion) show that there are no periodic orbits.

6.5.19 Consider the Volterra-Lotka predator prey model (rabbits and foxes although V-L wrote about fish in the Mediterranean)

$$\dot{R} = aR - bRF$$
$$\dot{F} = -cF + dRF$$

where $a, b, c, d > 0$ are constants.

i. Discuss the biological meaning of each of the terms in model. Comment on any unrealistic assumptions.

ii. Show that the model may be recast in dimensionless form

$$x' = x(1 - y)$$
$$y' = \mu y(x - 1)$$

iii. Find a conserved quantity in terms of the dimensionless variables.

iv. Show that the model predicts cycles in the populations of both species for almost all initial conditions.

6.6.5 Consider equations of the form

$$\ddot{x} + f(\dot{x}) + g(x) = 0$$

where f and g are smooth functions such that f is an even function.

i. Show that the equation is invariant under pure time-reversal symmetry $t \to -t$.

ii. Show that equilibrium points cannot be stable nodes or spirals.

Please hand in your problems G on Tuesday, February. 27.

G. Exercises from Sec. 7.1 – 7.3 of the text

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7.1[5, 8], 7.2[7, 12, 18], 7.3[1, 10]
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7.1.5 Show that the systems are equivalent where $x = r\cos\theta$, $y = r\sin\theta$

$$\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = 1 \end{cases} \qquad \begin{cases} \dot{x} = x - y - x(x^2 + y^2) \\ \dot{y} = x + y - y(x^2 + y^2) \end{cases}$$

7.1.8 Consider the equation for $a > 0$

$$\ddot{x} + a(\dot{x}^2 + x^2 - 1)\dot{x} + x = 0$$

(a) Find and classify all fixed points.
(b) Show that the system has a circular limit cycle. Find its amplitude and period.
(c) Determine the stability of the limit cycle.
(d) Give an argument that the limit cycle is unique, *i.e.,* there are no other periodic trajectories.

7.2.7 Consider

$$\begin{aligned}\dot{x} &= y + 2xy \\ \dot{y} &= x + x^2 - y^2\end{aligned}$$

(a) Show that $\partial f/\partial y = \partial g/\partial x$ so that the system is a gradient system.
(b) Find the potential V.
(c) Sketch the phase portrait.

7.2.12 Show that the system has no periodic trajectory

$$\begin{aligned}\dot{x} &= -x + 2y^3 - 2y^4 \\ \dot{y} &= -x - y + xy\end{aligned}$$

Hint: Find a, m, n such that $V = x^m + ay^n$ is a Liapunov function.

7.2.18 Consider Hofbauer and Sigmund's predator-prey model for $r > 0$ and $x \geq 0$, $y \geq 0$

$$\begin{aligned}\dot{x} &= rx\left(1 - \frac{x}{2}\right) - \frac{2xy}{1+x} \\ \dot{y} &= -y + \frac{2xy}{1+x}\end{aligned}$$

Prove that it has no periodic trajectory. Use Dulac's Criterion with $g(x,y) = \dfrac{1+x}{x}y^\alpha$ with suitable α.

7.3.1 Consider

$$\begin{aligned}\dot{x} &= x - y - x(x^2 + 5y^2) \\ \dot{y} &= x + y - y(x^2 + y^2)\end{aligned}$$

(a) Classify the fixed point at the origin.
(b) Rewrite the system in polar coordinates (r, θ).
(c) Determine the circle of maximal radius r_1 centered at the origin such that all the trajectories have a radially outward component on it.
(d) Determine the circle of minimal radius r_2 centered at the origin such that all the trajectories have a radially inward component on it.
(e) Prove that the system has a limit cycle somewhere in the annulus $r_1 \leq r \leq r_2$.

7.3.10 Consider the two-dimensional system

$$\dot{\mathbf{x}} = A\mathbf{x} - r^2\mathbf{x}$$

where $r = \|\mathbf{x}\|$ and A ia a constant 2×2 matrix with complex eigenvalues $\alpha \pm i\omega$. Prove that there exists at least one limit cycle if $\alpha > 0$ and that there are none for $\alpha < 0$.

Please hand in problems H on Tueday, Mar. 12.

H. Exercises from Sec. 8.1 – 8.3 of the text

`8.1[1, 4, 6, 13], 8.2[1], 8.3[1]`

8.1.1 Plot the phase portraits as μ varies.

$$(a.)\begin{cases}\dot{x} = \mu x - x^2 \\ \dot{y} = -y\end{cases} \qquad (b.)\begin{cases}\dot{x} = \mu x + x^3 \\ \dot{y} = -y\end{cases}$$

8.1.4 Find the eigenvalues at the stable fixed point as a function of μ, and show that one of the eigenvalues tends to zero as $\mu \to 0$.

$$\begin{aligned}\dot{x} &= \mu x + x^3 \\ \dot{y} &= -y\end{aligned}$$

8.1.6 Consider the system

$$\begin{aligned}\dot{x} &= y - 2x \\ \dot{y} &= \mu + x^2 - y\end{aligned}$$

(a) Find the nullclines.
(b) Find and classify all bifurcations that occur as μ varies.
(c) Sketch the phase portrait as a function of μ.

8.1.13 Recall the laser model of Exercise 3.3.1 for the number of excited atoms $N(t)$ and the number of photons $n(t)$

$$\begin{aligned}\dot{n} &= GnN - kn \\ \dot{N} &= -GnN - fN + p\end{aligned}$$

where f, G, k are positive constants and p is a constant of either sign.

(a) Nondimensionalize the system.
(b) Find and classify all of the fixed points.
(c) Sketch all the quantitatively different phase portraits that occur as the dimensionless parameters are varied.
(d) Plot the stability diagram for the system. What types of bifurcations occur?

8.2.1 Consider the biased van der Pol Equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$$

Find the curves in the (μ, a) space at which Hopf bifurcations occur.

8.3.1 Consider the Brusselator model of a chemical reaction for concentrations $x \geq 0$ and $y \geq 0$

$$\begin{aligned}\dot{x} &= 1 - (b+1)x + ax^2 y \\ \dot{y} &= bx - ax^2 y\end{aligned}$$

where $a > 0$ and $b > 0$ are parameters.

(a) Find all the fixed points, and use the Jacobian to classify them.

(b) Sketch the nullclines. Find as small as possible trapping region for the flow.

(c) Show that a Hopf bifurcation occurs at some parameter value $b = b_c$, where b_c is to be determined.

(d) Does the limit cycle exist for $b > b_c$ or $b < b_c$? Explain using the Poincaré Bendixson Theorem.

(e) Find the approximate period of the limit cycle for $b \approx b_c$.

Please hand in problems I on Tuesday, Mar. 19.

In grading your homework, I have recently noticed that too many homework solutions are almost identical to each other. The same notation, steps, tricks and shortcomings appear. I can only conclude that some students are copying from each other or copying solutions found on line word for word. It is bad enough that the source isn't attributed, but it is an example of claiming another's work as your own. This is plagiarism and is a violation of the University of Utah's Student Code (see our syllabus or Article XI). In the future, when I see similar solutions, I will not just mark them "Don't Copy," but will assign a zero score.

Nowadays the internet is omnipresent. I don't want to prevent students from accessing this useful resource or, once in a while, looking there for hints on a difficult problem. Similarly, I want to encourage students to work together and exchange ideas about problems, or to seek help from me or others. But, when it comes to writing the homework, students should work alone. In writing up solutions, students should use their own words and thereby make an original, intellectual contribution in explaining what they've worked hard to learn. Answers should be thorough enough to convince me you understand the solution and should attribute any extraordinary sources.

My apologies for this message to the majority of students who don't copy.

I. Exercises from Sec. 8.5 – 8.6 of the text

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8.5[3, 4], 8.6[1, 2]
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8.5.3 Consider the Logistic Equation with periodic carrying capacity

$$\dot{N} = rN\left(1 - \frac{N}{K(t)}\right)$$

where the carrying capacity $K(t) > 0$ is a smooth T-periodic function.

(a) Using a Poincaré Map argument, show that the system has at least one stable T-periodic limit cycle, contained in the strip $K_{\min} \le N(t) \le K_{\max}$.

(b) Is the cycle necessarily unique?

8.5.4 Consider a generalization of the fishery model of Exercise 3.7.3 where the harvesting is now a periodic function reflecting seasonal variations. Assume Benardete *et al.*'s model where the harvesting is sinusoidal. If fish population grows logistically in the absence of harvesting, the model is given in dimensionless form by

$$\dot{x} = rx(1 - x) - h(1 + \alpha \sin t)$$

where $r, h > 0$ and $0 < \alpha < 1$ are parameters.

(a) If $r < 4h$, show that there is no periodic solution even though the fish is being harvested with period $T = 2\pi$. What happens to the fish population in this case?

(b) Using a Poincaré Map argument, show that if $4h(1+\alpha) < r$, there exists a stable 2π-periodic solution in the strip $1/2 < x < 1$. Similarly there exists an unstable 2π-periodic solution in $0 < x < 1/2$. Interpret these results biologically.

(c) What happens in between cases (a) and (b) when $4h < r < 4h(1+\alpha)$?

8.6.1 Consider Ermentrout & Koppel's neural oscillator model with parameters $\omega_1, \omega_2 \geq 0$

$$\dot{\theta}_1 = \omega_1 + \sin\theta_1 \cos\theta_2$$
$$\dot{\theta}_2 = \omega_2 + \sin\theta_2 \cos\theta_1$$

(a) Sketch the qualitatively different phase portraits that arise when ω_1 and ω_2 vary.

(b) Find the curves in (ω_1, ω_2) parameter space along which bifurcations occur, and classify the various bifurcations.

8.6.2 Continuation of the previous problem. Now suppose

$$\dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1)$$
$$\dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2)$$

(a) Find a conserved quantity for the system. Hint: solve for $\sin(\theta_2 - \theta_1)$ in two ways.

(b) Suppose $K_1 = K_2$. Show that the system can be non-dimensioqnalized to

$$\frac{d\theta_1}{d\tau} = 1 + a\sin(\theta_2 - \theta_1)$$
$$\frac{d\theta_2}{d\tau} = \omega + a\sin(\theta_1 - \theta_2)$$

(c) Find the *winding number* ψ analytically, where

$$\psi = \lim_{\tau\to\infty} \frac{\theta_1(\tau)}{\theta_2(\tau)}$$

Hint: Evaluate the long time averages

$$\left\langle \frac{d(\theta_1 + \theta_2)}{d\tau} \right\rangle, \qquad \left\langle \frac{d(\theta_1 - \theta_2)}{d\tau} \right\rangle$$

where

$$\langle f \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\tau)\, d\tau.$$

Please hand in problems J on Tuesday, Mar. 26.

J Exercises from Sec. 8.7 – 9.1 of the text

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8.7[1, 2, 3], 9.1[3]
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8.7.1 Use partial fractions to evaluate the integral from Example 8.7.1 and use it to compute $r_1(r_0)$. Then check that $P'(r^*) = e^{-4\pi}$.

$$\int_{r_0}^{r_1} \frac{dr}{r(1-r^2)}, \qquad r_1 = \frac{1}{\sqrt{1 + e^{-4\pi}(r_0^{-2} - 1)}}$$

8.7.2 Consider the vector field on the cylinder

$$\dot{\theta} = 1$$
$$\dot{y} = ay$$

Define an appropriate Poincaré Map and find a formula for it. Show that the system has a periodic orbit. Classify its stability for all real values of a.

8.7.3 Consider an over damped linear oscillator forced by a square wave. The system can be non-dimensionalized to

$$\dot{x} + x = F(t)$$

where

$$F(t) = \begin{cases} +A, & \text{if } 0 \le t < \frac{T}{2}, \\ -A, & \text{if } \frac{T}{2} \le t < T \end{cases}$$

for all $0 \le t < T$, and the $F(t)$ is extended by T-periodicity to all other t. Show that trajectories approach a unique periodic solution.

(a) Let $x(0) = x_0$. Show that $x(T) = x_0 e^{-T} - A(1 - e^{-T/2})^2$.

(b) Show that the system has a unique periodic solution that satisfieds $x_0 = -A \tanh(T/4)$.

(c) Interpret the limits of $x(T)$ as $T \to \infty$. Explain why they're plausible.

(d) Let $x_1 = x(T)$ and define the Poincaré Map P as $P(x_0) = x_1$. More generally $x_{k+1} = P(x_k)$. Plot the graph of P.

(e) Using the cobweb picture, shew what P has a globally stable fixed point.

9.1.3 Find a change of variables that converts the water wheel equations

$$\dot{a}_1 = \omega b_1 - K a_1$$
$$\dot{b}_1 = -\omega a_1 + q_1 - K b_1$$
$$\dot{\omega} = -\frac{\nu}{I}\omega + \frac{\pi g r}{I} a_1$$

into the Lorenz equations

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - xz - y$$
$$\dot{z} = xy - bz$$

where $\sigma, b, r > 0$ are parameters. Also show that when the waterwheel equations are transformed into the Lorenz equations, then the Lorenz parameter b turns out to be $b = 1$. Express the Prandtl and Rayleigh numbers σ and r in terms of the waterwheel parameters.

Please hand in problems K on Tuesday, Apr. 2.

K Exercises from Sec. 9.1 – 9.2 of the text

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9.1[4], 9.2[3, 6]
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9.1.4 The Maxwell-Bloch equations for a laser are

$$\dot{E} = \kappa(P - E)$$
$$\dot{P} = \gamma_1(ED - P)$$
$$\dot{D} = \gamma_2(\lambda + 1 - D - \lambda EP)$$

(a) Show that the non-lading state (the fixed point with $E^* = 0$) loses stability above a threshold value of λ, to be determined. Classify the bifurcation at this laser threshold.

(b) Find the change of variables that transforms this system into the Lorenz system.

9.2.3 Show that all trajectories of the Lorenz equations eventually enter and remain inside the sphere

$$x^2 + y^2 + (z - r - \sigma)^2 = C$$

for C sufficiently large. [Hint: show that $x^2+y^2+(z-r-\sigma)^2$ decreases on trajectories outside a certain ellipsoid. Then pick C large enough so that the sphere engulfs this ellipsiod.]

9.2.6 Consider Rikitake's model for geommagnetic reversals with $a, \nu > 0$ parameters

$$\begin{aligned} \dot{x} &= -\nu x + zy \\ \dot{y} &= -\nu y + (z - a)x \\ \dot{z} &= 1 - xy \end{aligned}$$

(a) Show that the system is dissipative.

(b) Show that the fixed points may be written $x^* = \pm k$, $y^* = \pm k^{-1}$, $z^* = \nu k^2$ where $\nu(k^2 - k^{-2}) = a$.

(c) Classify the fixed points.

Please hand in problems L1 – L2 on Tuesday, Apr. 9.

L1 Exercises from Sec. 9.3 – 10.1 of the text

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9.3[8], 9.4[2], 10.1[10]
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9.3.8 Consider the following system in polar coordinates. $\dot{r} = r(1 - r^2)$, $\dot{\theta} = 1$.
Let $D = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\}$ be the closed unit disk.

(a) Is D an invariant set?

(b) Does D attract an open set of initial conditions?

(c) Is D an attractor? If not, why not? If so, find its basin of attraction.

(d) Repeat part (c) for the circle $C = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$.

9.4.2 Consider the map

$$x_{n+1} = \begin{cases} 2x_n, & \text{if } 0 \leq x_n \leq \frac{1}{2}, \\ 2 - 2x_n, & \text{if } \frac{1}{2} < x_n \leq 1. \end{cases}$$

(a) Why is it called a "tent map"?

(b) Find all the fixed points and classify their stability.

(c) Show that the map has a period-2 obit. Is it stable or unstable?

(d) Can you find any period-3 points? How about period-4? If so, are the corresponding periodic orbits stable or unstable?

10.1.10 Show that the map $x_{n+1} = 1 + \frac{1}{2}\sin x_n$ has a unique fixed point. Is it stable?

L2. Proposals for term projects are due Apr. 9. Meet with me briefly to discuss your project. Bring a one paragraph proposal of the topic you will write about for my approval.

In your term paper, you will discuss some current or historical theory or application of ordinary differential equations or chaos that is not covered in the course, but it should be at the level of the course. It should be a five page mathematical paper written at a level appropriate for Math 5470 students. It should focus on one system of equations. If it is a theoretical or historical paper, it should contain some theorems with proofs that describe some phenomenon. If it is an applied paper, it should develop the equations from first principles, analyze them using tools from the course and draw conclusions about the application from the analysis. The paper should be written in proper English style, following AMS, APA, MLA, or other recognized guidelines. Please get in touch with me if you'd like advice on your topic.

Your proposal should be a one paragraph description of what you will be writing about. Please include a tentative title. In addition to a description of the equation and what you want to say about it, please include at least one reference to the topic from a book or scholarly article and a reference from the internet. Please include the URL of any website and track down its author. Indicate the which style guideline you'll follow.

I will approve any reasonable proposal. The main reason I have objected is to proposals is that they propose to do more than is possible in five pages, which is a very short paper.

Please hand in problems M1 on Tuesday, Apr. 16.

M1 Exercises from Sec. 9.4 – 11.2 of the text

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10.1[12], 10.3[2], 11.2[1]
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10.1.12 To find the roots of the equation $g(x) = 0$, Newton's Method is to iterate the map $x_{n+1} = f(x_n)$, where

$$f(x_n) = x_n = \frac{g(x_n)}{g'(x_n)}.$$

(a) To calibrate the method, write down the Newton Map $x_{n+1} = f(x_n)$ for the equation $g(x) = x^2 - 4 = 0$.

(b) Show that the Newton Map has fixed points $x^* = \pm 2$,

(c) Show that the fixed points are superstable.

(d) Iterate the map numerically, startig from $x_0 = 1$.

10.3.2 Let p and q be points in a 2-cycle for the logistic map.

(a) Show that if the cycle is superstable then either $p = \frac{1}{2}$ or $q = \frac{1}{2}$.

(b) Find the value r at which the logistic map has a superstable 2-cycle.

11.2.1 Here is another way to show that the Cantor set has measure zero. In the first stage of construction we removed an interval of length $\frac{1}{3}$ from the unit interval $[0, 1]$. At the next stage we removed two intervals each of length $\frac{1}{9}$. By summing the appropriate infinite series, show that the total length removed is 1, and hence the leftovers (the Cantor Set) have length zero.

Please hand in **Term Projects** on Tuesday, Apr. 23. The **Final** is Thursday, Apr. 25 from 10:30 am – 12:30 ap.